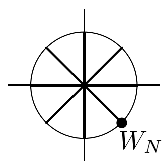
The Discrete Fourier Transform (DFT) converts a signal from the time representation $s_n = s_0, s_1, \ldots s_{N-1}$ to the frequency representation $S_k = S_0, S_1, \ldots S_{N-1}$. It is given by

$$S_k = \sum_{n=0}^{N-1} s_n W_N^{nk}$$

where $k = 1, \ldots, N-1$.

In this expression we used the N^{th} *root of unity*, i.e., the complex number $W_N = e^{-i\frac{2\pi}{N}} = \cos(\frac{2\pi}{N}) - i\sin(\frac{2\pi}{N})$ which gives 1 when raised to the N^{th} power ($W_N^N = 1$). The geometric interpretation of W_N is obtained by dividing the unit circle into N equal slices, and marking the end of the first slice in the clockwise (negative) direction.



How do we convert back from the frequency domain representation S_k to the time domain representation s_n? This is what the inverse Discrete Fourier Transform, the iDFT, does

$$s_n = \frac{1}{N} \sum_{k=0}^{N-1} S_k W_N^{-nk}$$

where it is merely a (bad) convention for the $\frac{1}{N}$ normalization to be in the iDFT rather than in the DFT.

To see that this formula is correct, we need to see that the iDFT of the DFT returns the original signal s_n. This follows from one of the trigonometric *orthogonality* rules

$$\sum_{k=0}^{N-1} W_N^{n_1 k} W_N^{-n_2 k} = N\delta_{n_1,n_2}$$

that is, the norm of a frequency domain vector W_N^{nk} is N and the dot product of two *different* vectors $W_N^{n_1 k}$ and $W_N^{n_2 k}$ is zero.

> For those interested - how can we prove this orthogonality rule? When $n_1 = n_2$ we have
>
> $$\sum_{k=0}^{N-1} W_N^{n_1 k} W_N^{-n_1 k} = \sum_{k=0}^{N-1} 1$$
>
> which is obviously N. When $n_1 \neq n_2$ their difference is some integer $m = n_1 - n_2$
>
> $$\sum_{k=0}^{N-1} W_N^{n_1 k} W_N^{-n_2 k} = \sum_{k=0}^{N-1} W_N^{mk} = \sum_{k=0}^{N-1} (W_N^m)^k$$

which is a geometric series with ratio $q = W_N^m$. Summing we find

$$\sum_{k=0}^{N-1} q^k = \frac{q^N - 1}{q - 1}$$

but $q^N = 1$ since q is a power of W_N which is an N^{th} root of unity, so the numerator is zero (and since $|q| = 1$ we don't have to worry about the denominator being zero).

Substituting

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{n'=0}^{N-1} s_{n'} W_N^{n'k} W_N^{-nk} = \frac{1}{N}\sum_{n'=0}^{N-1} s_{n'} \sum_{k=0}^{N-1} W_N^{n'k} W_N^{-nk} = \frac{1}{N}\sum_{n'=0}^{N-1} s_{n'} N \delta_{n',n} = s_n$$

and it is similar to show that the DFT of the iDFT returns S_k.

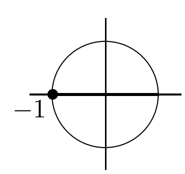
It is often easier to write the DFT in matrix notation.

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ 1 & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)^2} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_{N-1} \end{pmatrix}$$

Notice that the first row and first column are always all ones since $W_N^0 = 1$. The iDFT can also be written in matrix form.

Let's see two simple examples.

For $N = 2$ it is easy to see that $W_2 = e^{-i\frac{2\pi}{2}} = e^{-i\pi} = \cos(-\pi) = -1$ which is logical since $-1^2 = 1$ and so -1 is the square root of unity. In the geometric interpretation we divide the circle into an upper and lower half, 180 degrees each, and -1 is obtained.



Substituting into the formula for the DFT we readily find

$$\begin{aligned} S_0 &= \sum_{n=0}^{1} s_n W_2^0 = s_0 + s_1 \\ S_1 &= \sum_{n=0}^{1} s_n W_2^n = s_0 - s_1 \end{aligned}$$

which has a simple interpretation. The zeroth (DC) coefficient is simply the sum (i.e., twice the average of s_0 and s_1). The other (high-frequency) coefficient is the difference (the derivative).

In matrix notation

$$\begin{pmatrix} S_0 \\ S_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix}$$

What about the iDFT for $N = 2$?

$$\begin{aligned} s_0 &= \frac{1}{2}\sum_{k=0}^{1} s_n W_2^0 = \frac{1}{2}(S_0 + S_1) \\ s_1 &= \frac{1}{2}\sum_{k=0}^{1} s_n W_2^{-n} = \frac{1}{2}(S_0 - S_1) \end{aligned}$$

or

$$\begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}$$

in matrix notation.

We can test that this is correct!

$$\begin{aligned} \frac{1}{2}\left\{(s_0 + s_1) + (s_0 - s_1)\right\} = s_0 \\ \frac{1}{2}\left\{(s_0 + s_1) - (s_0 - s_1)\right\} = s_1 \end{aligned}$$

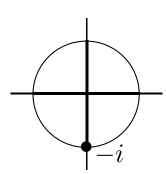
In matrix notation note that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

so that putting the $\frac{1}{2}$ in either matrix gives us the unity matrix.

Now let's find the DFT for $N = 4$!

For $N = 4$ we find $W_4 = e^{-i\frac{2\pi}{4}} = e^{-i\frac{\pi}{2}} = \sin(-\frac{\pi}{2}) = -i$ which is logical since $-i^4 = i^4 = (-1)^2 = 1$ and so $-i$ is the fourth root of unity. In the geometric interpretation we divide the circle into four slices, 90 degrees each.



Now we find

$$
\begin{aligned}
S_0 &= \sum_{n=0}^{1} s_n W_4^0 = s_0 + s_1 + s_2 + s_3 \\
S_1 &= \sum_{n=0}^{1} s_n W_4^n = s_0 + (-i)s_1 + (-i)^2 s_2 + (-i)^3 s_3 \\
S_2 &= \sum_{n=0}^{1} s_n W_4^{2n} = s_0 + (-i)^2 s_1 + (-i)^4 s_2 + (-i)^6 s_3 \\
S_3 &= \sum_{n=0}^{1} s_n W_4^{3n} = s_0 + (-i)^3 s_1 + (-i)^6 s_2 + (-i)^9 s_3
\end{aligned}
$$

and it is left as an exercise to reduce the powers and find the explicit coefficients.

We can write this in matrix format as well

$$
\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & (-i)^2 & (-i)^3 \\ 1 & (-i)^2 & (-i)^4 & (-i)^6 \\ 1 & (-i)^3 & (-i)^6 & (-i)^9 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}
$$

and I leave it to you to fill in the matrix.

Finally, it is instructive to find the iDFT for this case and mutliply the two 4*4 matrices to make sure they give the identity matrix.