2.3.1 Look closely at the graphs of the digital sinusoid $s_n = \sin(\omega n)$ that you prepared in exercise 2.2.2. When is the digital sinusoid periodic? Under what conditions is the period the same as that of the analog sinusoid?

An analog sinusoid is always periodic, since $\sin(\omega(t+\tau))$ is equal to $\sin(\omega t)$ when $\omega\tau = 2\pi n$ for any integer n. Thus, after $\tau = \frac{2\pi n}{\omega}$, the analog sinusoid returns to the same value.

After plotting a few digital signals you can see that this is not always the case. For example, if $\omega = 1$ then $s_n = \sin(\omega n) = \sin(n)$, i.e., $s_0 = 0$, $s_1 = \sin(1) \approx 0.841$, $s_2 = \sin(2) \approx 0.909$, and thereafter 0.141, -0.757, -0.959, -0.279, 0.657, 0.989, etc. We never seem to return to the same value!

To prove that this is true assume that at some large value n, $\sin(n)$ returns to $\sin(0) = 0$. Since we know that $\sin(\theta) = 0$ only for $\theta = k\pi$, we can deduce from $\sin(n) = 0$ that $n = k\pi$ and thus that $\pi = \frac{n}{k}$. But π is irrational, and thus this can't be true.

Reasoning positively, for $\sin(\omega n)$ to be periodic, we must have $\sin(\omega(n+N)) = \sin(\omega n)$, which implies that $\omega(n+N) = \omega n + 2\pi k$. Hence, $\omega N = 2\pi k$, and $\omega = \frac{2\pi k}{N} = \frac{2k}{N}\pi$. So the condition is that the ω be a rational multiple of π.

We can think about this is another way. Start with the periodic analog sinusoid $s(t) = \sin(\omega t) = \sin(2\pi f t)$, and sample it to obtain the digital one. If we sample each analog sine period some whole number of times (say N times per period), then obviously the digital sinusoid will be periodic with period N. So, if the sampling rate is an integer multiple of $2\pi f$, i.e., $2\pi N f$, then the digital sinusoid will be periodic.

In this case the the digital period is 'the same' as the analog period, i.e., the digital period is the analog period divided by the sampling interval.

But, what if we under-sample the analog sinusoid? Remember, we are only asking about periodicity, and don't care if we have the ability to reproduce the original signal (and so the sampling theorem is not important here). If we sample exactly once every Nth period (say exactly at the top, but it could be anywhere on the sine), we obviously obtain a periodic digital signal, with period of 1. So, sampling an analog sinusoid at a rate of $\frac{1}{N}$ also yields a periodic digital signal.

Finally, we can combine these two cases. For example, if we sample every two analog periods exactly three times (i.e., a sampling rate of $\frac{2}{3}$, or every 17 analog periods exactly 137 times, we obtain a periodic digital sinusoid. So, the condition can be worded - sample any analog sinusoid a rational number of times per period.

This is the same condition as our first one. Do you see why?