2.4.1 Show that the exponential signal $s_n = Ae^{\Lambda n}$ is an eigensignal of the time *advance* operator. What is its eigenvalue?

An operator usually tranforms a signal into a profoundly different signal, but eigensignals are not fundamentally changed, they merely acquire a trivial gain.

In other words, an eigensignal s_n of an operator $\hat{O}$ obeys

$$\hat{O}\, s_n = \lambda s_n$$

where λ, the *eigenvalue*, is a constant real number.

The time advance operator $\hat{z}$ is defined as the operation of returning *now* the value the signal will be next time

$$\hat{z}\, s_n = s_{n+1}$$

which usually requires a crystal ball.

Of course, for deterministic signals we know what the signal's value will be for any time, and there is no problem with predicting its value at time $n + 1$. However, for stochastic signals, or signals not under our complete control, the time advance operator is not implementable in real time, and for this reason we usually work with the time delay operator $\hat{z}^{-1}$.

For deterministic signals it is easy to apply the time advance operator; we need simply replace n with $n + 1$.

$$\hat{z}\, s_n = s_{n+1} = Ae^{\Lambda(n+1)} = Ae^{\Lambda n+\Lambda} = Ae^{\Lambda}e^{\Lambda n} = e^{\Lambda}(Ae^{\Lambda n}) = e^{\Lambda}s_n$$

so $s_n = Ae^{\Lambda n}$ is indeed an eigensignal of $\hat{z}$ and the eigenvalue is e^{Λ}.

The real sinusoid $s_n = A\sin(\omega n+\phi)$ is the eigensignal of an operator that contains z^{-1} and z^{-2}. Can you find this operator?

The question is in principle similar to the first part, but somewhat more challenging from a technical point of view. We are asked to find an operator that contains two time delays; we will assume the simplest combination, namely a weighted linear combination.

$$\hat{O} \;=\; az^{-1} + bz^{-2}$$

What does this combination do to a signal s_n?

$$\hat{O}\, s_n \;=\; (a\hat{z}^{-1} + b\hat{z}^{-2})s_n = a\hat{z}^{-1}s_n + b\hat{z}^{-2}s_n = as_{n-1} + bs_{n-2} \qquad (1)$$

Now we are interested in applying this operator to a digital sinusoid

$$s_n = A\sin(\omega n + \phi)$$

and finding values a and b such that

$$s_n \; = \; a s_{n-1} + b s_{n-2} \tag{2}$$

that is, for which the following holds.

$$A\sin(\omega n + \phi) \; = \; a\,A\sin(\omega(n-1)+\phi) \;+\; b\,A\sin(\omega(n-2)+\phi)$$

The straight forward way of finding *if* this is can be true, and for which a and b involves using a lot of trigonometric identities.

$$\begin{aligned}
s_{n-1} &= A\sin(\omega(n-1)+\phi) \\
&= A\sin((\omega n+\phi)-\omega) \\
&= A\sin(\omega n+\phi)\cos(\omega) - A\cos(\omega n+\phi)\sin(\omega) \\
s_{n-2} &= A\sin(\omega(n-2)+\phi) \\
&= A\sin((\omega n+\phi)-2\omega) \\
&= A\sin(\omega n+\phi)\cos(2\omega) - A\cos(\omega n+\phi)\sin(2\omega) \\
&= A\sin(\omega n+\phi)(2\cos^2(\omega)-1) - A\cos(\omega n+\phi)(2\sin(\omega)\cos(\omega)) \\
&= 2\cos(\omega)\,[A\sin(\omega n+\phi)\cos(\omega) - A\cos(\omega n+\phi)\sin(\omega)] - A\sin(\omega n+\phi)
\end{aligned}$$

Now, the expression in the square brackets on the last line is precisely what we found for s_{n-1} and the final term is s_n itself. So we have found $s_{n-2} = 2\cos(\omega)s_{n-1} - s_n$ or

$$s_n \; = \; 2\cos(\omega)s_{n-1} - s_{n-2} \tag{3}$$

and by comparing this to 2 we see that indeed all digital sinusoids obey a second order *difference equation*, and that $a = 2\cos(\omega)$ and $b = -1$. Using precisely the same manipulations, the sinusoid's equation is found for analog sinusoids on page 249 of the book.

The above proof is straightforward, but somewhat tiring. There are several ways to simplify the algebra. One way is to realize that equation 1 can be rewritten

$$s_{n+1} = a s_n + b s_{n-1}$$

without changing its meaning. This time we will assume $\phi = 0$ in order to further simplify.

$$A\sin(\omega(n+1)) \; = \; a\,A\sin(\omega n) \;+\; b\,A\sin(\omega(n-1))$$

Now the left hand side is

$$A\sin(\omega n)\cos(\omega) + A\cos(\omega n)\sin(\omega)$$

and the right hand side is only slight more complex.

$$a\,A\sin(\omega n)\;+\;b\,(A\sin(\omega n)\cos(\omega)-A\cos(\omega n)\sin(\omega))$$

By comparing the coefficient of $\cos(\omega n)$ on both sides we find $b=-1$, and then comparing the coefficients of $\sin(\omega n)$ leads us directly to $a=2\cos(\omega)$.

Another way of going about this is to use complex exponentials. As usual, this significantly simplifies the algebra as we don't need difficult trigonometric identities.

$$e^{\mathrm{i}\omega n}=ae^{\mathrm{i}\omega(n-1)}+be^{\mathrm{i}\omega(n-2)}=\left(ae^{-\mathrm{i}\omega}+be^{-2\mathrm{i}\omega}\right)e^{\mathrm{i}\omega n}$$

For this to hold,

$$ae^{-\mathrm{i}\omega}+be^{-2\mathrm{i}\omega}=1$$

but note that trivially

$$(e^{\mathrm{i}\omega}+e^{-\mathrm{i}\omega})e^{-\mathrm{i}\omega}-e^{-2\mathrm{i}\omega}=1$$

so $a=(e^{\mathrm{i}\omega}+e^{-\mathrm{i}\omega})=2\cos(\omega)$ and $b=-1$.

Our last solution will be a bit more elegant. Observe what the time advance and time delay operators do to a generic sinusoid.

$$\begin{aligned}\hat{z}^{+1}\sin(\omega n+\phi) &= \sin(\omega n+\phi)\cos(\omega)+\cos(\omega n+\phi)\sin(\omega)\\ \hat{z}^{-1}\sin(\omega n+\phi) &= \sin(\omega n+\phi)\cos(\omega)-\cos(\omega n+\phi)\sin(\omega)\end{aligned}$$

We can define an operator to be the average of the time delay and time advance operators, that is, $\hat{\zeta}=\frac{1}{2}(\hat{z}^{-1}+\hat{z}^{+1})$. Based on the above equations, when this operator is applied to a generic sinusoid, the second terms above cancel.

$$\hat{\zeta}\sin(\omega n+\phi)=\cos(\omega)\sin(\omega n+\phi)\tag{4}$$

Therefore $s_n=\sin(\omega n+\phi)$ is an eigensignal of the operator $\hat{\zeta}$ with eigenvalue $\cos(\omega)$! The solution to the exercise is now immediate.