2.4.3 Compare the energy of a time delayed, advanced, or reversed signal with that of the original signal. What is the energy of $y = ax$ in terms of that of x? What can be said about the energy of the sum of two signals? For example, consider summing two sinusoids of the same frequency but different amplitudes and phases. What about two sinusoids of different frequencies? Why is there a difference between these two cases?

Energy represents the *size* of a signal, or more importantly the *cost* of producing a signal. Thus it is intuitively obvious that the energies of time delayed, advanced, or reversed signals must be the same as that of the original signal. Time delay and advance can not change the energy as they are equivalent to merely redefining what we consider to be time *zero*, which can not change the energy.

Time reversal is trickier to justify, but still obvious when we think of energy as *size*. The mathematical demonstration derives from the form of the expression for energy, that is, from the fact that the energy is a sum over all the values squared. Sums can be performed in either direction from minus infinity up to plus infinity, or from plus infinity down to minus infinity. The change in order has no relevance for convergent sums of positive terms.

On the other hand, multiplying a signal by a gain definitely changes the signal's *size*. It is important to understand why this question, and the question of the energy of the sum of two signals, is interesting. The proof that the set of signals is a linear vector space relies on the fact that the amplified signal, and the sum of two signals, are themselves *signals*. Since finite energy is a requirement for an entity to be true signal, we need to be sure that the amplification or sum produces something with finite energy.

The multiplication of a signal by a number does not always change the energy. If the gain is one, or minus one, the square of the values is unchanged, and thus the energy is unchanged. If the *gain* a is in fact an *attenuation*, i.e., if $|a| < 1$, the signal is smaller in size, and the energy should decrease. These facts together lead us to believe that if the energy of the original signal x is E_x, then the energy of $y = ax$ should be $E_y = a^2 E_x$. Let's see that this is the case. For an analog signal

$$E_y = \int y^2(t)dt = \int (ax(t))^2 dt = \int a^2 x^2(t)dt = a^2 \int x^2(t)dt = a^2 E_x$$

as expected; while for a digital signal

$$E_y = \sum y_n^2 = \sum (ax_n)^2 = \sum a^2 x_n^2 dt = a^2 \sum x_n^2 dt = a^2 E_x$$

where the integral and sum are over all times.

However, when we look at the sum of two signals $s_n = x_n + y_n$ (let's limit ourselves to digital signals). the situation is more complex.

$$\begin{aligned} E_s &= \sum (x_n + y_n)^2 = \sum \left(x_n^2 + 2x_n y_n + y_n^2 \right) \\ &= \sum x_n^2 + \sum y_n^2 + 2 \sum x_n y_n = E_x + E_y + 2C_{xy} \end{aligned}$$

where we have defined C_{xy}, the correlation between the two signals x and y, which is discussed in chapter 9. We don't yet know very much about the correlation yet, but note that this quantity can be zero, positive or negative. So the energy can be less than, equal to, or more than the sum of the energies.

Furthermore, for uncorrelated signals, i.e., for signals for which $C_{xy} = 0$, the energy of the sum is the sum of the energies. We will see in chapter 9 that signals that look similar are correlated, for example, if $y = ax$ then x and y are very correlated. On the other hand uncorrelated signals are signals that do not look anything like each other. For example, two sinusoids of different frequencies are completely uncorrelated. Don't be confused – they don't look similar because at a time when x is positive, the probability of y being positive too is only 50 percent.

So, for uncorrelated signals the energy of the sum is the sum of the energies, while for correlated signals the energy of the sum may be less than or more than the sum of the energies.

However, we may be able to say more. Let's limit ourselves to sinusoids. Of course, sinusoids have infinite energy, and thus the question is meaningless unless we do our regular trick of zeroing out the signals far enough in the past and future, i.e.,

$$x_n = \begin{cases} 0 & n < -N \\ A_x \sin(\omega_x n + \varphi_x) & -N \leq n \leq +N \\ 0 & n > +N \end{cases}$$

and similarly for y

$$y_n = \begin{cases} 0 & n < -N \\ A_y \sin(\omega_y n + \varphi_y) & -N \leq n \leq +N \\ 0 & n > +N \end{cases}$$

Although it is possible to find the energies E_x and E_y explicitly, all that we need to know is that this energy depends on the amplitude A, (any dependence on ω and φ being an artifact of the precise method of limiting the signal duration).

If we limit ourselves to the case when the two sinusoids have the same energy, i.e., $A_x = A_y$, but the frequencies may differ, we can nicely bound the energy of the sum. To see how, first note that while energy always obeys $E \geq 0$, it might be case that the energy of a sum may never be zero. But this is not the case here. It is clear that if $y = -x$, then according to the previous section of the exercise $E_y = E_x$, and thus this is a possibility here. In this case $s = x + y = 0$ the zero signal, and thus has zero energy.

On the other hand, if $y = x$, then $s = x + y = 2x$, and thus according to the previous section $E_s = 2^2 E_x = 4E_x$.

You may have guessed that $0 \leq E_{x+y} = 4E_x$, but we haven't proven that the sum may not have energy greater than $4E_x$. In order to see this, note that the energy of the difference between two sinusoids of equal energy always obeys

$$0 \leq E_{x-y} = \sum (x_n - y_n)^2 = \sum \left(x_n^2 + y_n^2 - 2x_n y_n == E_x + E_x - 2 \sum x_n y_n \right)$$

and therefore, but simple algebra we see

$$-2E \leq -2 \sum x_n y_n$$

which means

$$\sum x_n y_n \leq E$$

Plugging this in to the main equation,

$$0 \leq E_{x+y} = E_x + E_x + 2 \sum x_n y_n \leq E_x + E_x + 2E_x = 4E_x$$

as expected.

Now, what happens when $E_x \neq E_y$? In this case the energy can never be zero, since only the zero signal has zero energy. If the sinusoids have different frequencies then we know that $E_s = E_x + E_y$. So let's look at sinusoids with different energies but the same frequencies. When the signals are in phase, i.e., $x_n = A_x \sin(\omega n + \varphi)$ and $y_n = A_y \sin(\omega n + \varphi)$, then $s_n = x_n + y_n = (A_x + A_y) \sin(\omega n + \varphi)$ and the energy depends on $(A_x + A_y)^2$ while when they are 180 degrees out of phase, i.e., $x_n = A_x \sin(\omega n + \varphi)$ and $y_n = -A_y \sin(\omega n + \varphi)$, then $s_n = x_n + y_n = (A_x - A_y) \sin(\omega n + \varphi)$ and the energy depends on $(A_x - A_y)^2$

We can nail this down by looking at the correlation. When the signals are in phase,

$$y_n = \frac{A_y}{A_x} x_n = \sqrt{\frac{E_y}{E_x}} x_n$$

the correlation is

$$C_{xy} \equiv \sum x_n y_n = \sum x_n \left(\sqrt{\frac{E_y}{E_x}} x_n \right) = \sqrt{\frac{E_y}{E_x}} \sum x_n^2 = \sqrt{\frac{E_y}{E_x}} E_x = \sqrt{E_y E_x}$$

so that

$$E_s = E_x + E_y + 2\sqrt{E_y E_x}$$

and when the signals are out of phase

$$y_n = -\frac{A_y}{A_x} x_n = -\sqrt{\frac{E_y}{E_x}} x_n$$

the correlation is

$$C_{xy} = -\sqrt{E_y E_x}$$

so that

$$E_s = E_x + E_y - 2\sqrt{E_y E_x} \quad .$$

We thus reach the conclusion that

$$E_x + E_y - 2\sqrt{E_y E_x} \quad \leq \quad E \quad \leq \quad E_x + E_y + 2\sqrt{E_y E_x}$$

which trivially holds true for the case of different frequencies as well.