6.10.17 $\mathbf{a_0x_n + a_1x_{n-1}}$ can be written $\mathbf{A_0x_n + A_1\Delta x_n}$ where $\mathbf{a_1 = -A_1}$ and $\mathbf{a_0 = A_0 + A_1}$. What is the connection between the coefficients of $\mathbf{a_0x_n + a_1x_{n-1} + a_2x_{n-2}}$ and $\mathbf{A_0x_n + A_1\Delta x_n + A_2\Delta^2x_n}$? What about $\mathbf{a_0x_n + a_1x_{n-1} + a_2x_{n-2} + a_3x_{n-3}}$ and $\mathbf{A_0x_n + A_1\Delta x_n + A_2\Delta^2x_n + A_3\Delta^3x_n}$? Generalize and prove that all ARMA equations can be expressed as difference equations.

First, let's prepare a list of formulas for the finite differences.

$$
\begin{array}{lll}
(\Delta^0 x)_n & & = x_n \\
(\Delta^1 x)_n & = (\Delta^0 x)_n - (\Delta^0 x)_{n-1} & = x_n - x_{n-1} \\
(\Delta^2 x)_n & = (\Delta^1 x)_n - (\Delta^1 x)_{n-1} & = x_n - 2x_{n-1} + x_{n-2} \\
(\Delta^3 x)_n & = (\Delta^2 x)_n - (\Delta^2 x)_{n-1} & = x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} \\
(\Delta^4 x)_n & = (\Delta^3 x)_n - (\Delta^3 x)_{n-1} & = x_n - 4x_{n-1} + 6x_{n-2} - 4x_{n-3} + x_{n-4} \\
(\Delta^k x)_n & \quad = (1 - \mathrm{z}^{-1})^k x_n & = \displaystyle\sum_{i=0}^{k} (-1)^i \binom{k}{i} x_{n-i}
\end{array}
$$

and so on.
Now we can try the specific cases.

$$
\begin{aligned}
&A_0x_n + A_1\Delta x_n + A_2\Delta^2 x_n = \\
&A_0x_n + A_1\left(x_n - x_{n-1}\right) + A_2\left(x_n - 2x_{n-1} + x_{n-2}\right) = \\
&\left(A_0 + A_1 + A_2\right)x_n - \left(A_1 + 2A_2\right)x_{n-1} + A_2x_{n-2}
\end{aligned}
$$

from which we deduce

$$
\begin{aligned}
a_0 &= A_0 + A_1 + A_2 \\
a_1 &= -(A_1 + 2A_2) \\
a_2 &= A_2
\end{aligned}
$$

which can be written in matrix form.

$$
\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix}
$$

These formulas enable us to convert every difference equation of order 2

$$
y_n = A_0x_n + A_1\Delta x_n + A_2\Delta^2 x_n
$$

into an equivalent ARMA filter

$$y_n = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2}$$

by telling us how to find a_0, a_1, a_2 given A_0, A_1, A_2.
Unfortunately, the exercise asks for us to do precisely the opposite, namely to convert ARMA filters into difference equations, i.e. to express a_0, a_1, and a_2 in terms of A_0, A_1 and A_2. To do this we have to solve the above equations (which is simple by substitution from A_2 backwards to A_0) or equivalently to invert the above matrix (which is simple due to the matrix being upper triangular).

$$\begin{aligned} A_0 &= a_0 + a_1 + a_2 \\ A_1 &= -(a_1 + 2a_2) \\ A_2 &= a_2 \end{aligned}$$

Remarkably the equations for the A are of the same form as those for the a. The second order case was relatively trivial, so let's find the formulas for the third order case.

$$\begin{aligned} A_0 x_n + A_1 \Delta x_n + A_2 \Delta^2 x_n \quad & +A_3 \Delta^3 x_n = \\ A_0 x_n + A_1 (x_n - x_{n-1}) \quad & +A_2 (x_n - 2x_{n-1} + x_{n-2}) \\ & +A_3 (x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3}) = \\ (A_0 + A_1 + A_2 + A_3) x_n \quad & - (A_1 + 2A_2 + 3A_3) x_{n-1} \\ & + (A_2 + 3A_3) x_{n-2} - A_3 x_{n-3} \end{aligned}$$

i.e.

$$\begin{aligned} a_0 &= A_0 + A_1 + A_2 + A_3 \\ a_1 &= -(A_1 + 2A_2 + 3A_3) \\ a_2 &= A_2 + 3A_3 \\ a_3 &= -A_3 \end{aligned}$$

or in matrix form.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Once again the matrix is triangular and the equations can be solved by substitution from the last back towards the begining. Solving we find

$$\begin{aligned}
A_0 &= a_0 + a_1 + a_2 + a_3 \\
A_1 &= -(a_1 + 2a_2 + 3a_3) \\
A_2 &= a_2 + 3a_3 \\
A_3 &= -a_3
\end{aligned}$$

once again of the same form as the original equations. We have proved that we can convert any ARMA filter of order 3 into difference equation form.
It would be straightforward for the reader to verify that for the fourth order case the following holds.

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix}$$

Once again the matrix is upper triangular, and once again it is its own inverse. This time the matrix is large enough to start revealing its secrets. Disregarding the minus signs this matrix is simply Pascal's triangle transposed, with the familiar binomial coefficients running down its columns.
We can jump now to the general case. Since we have

$$\begin{aligned}
\sum_k A_k \Delta^k x_n &= \sum_k A_k (1 - \mathrm{z}^{-1})^k x_n \\
&= \sum_k A_k \sum_{i=0}^{k} (-1)^i \binom{k}{i} x_{n-i} \\
&= \sum_i \sum_k (-1)^i \binom{k}{i} A_k x_{n-i}
\end{aligned}$$

(defining the binomial coefficient to be zero when $i > k$), we can identify the coefficients.

$$a_i = \sum_k (-1)^i \binom{k}{i} A_k \tag{1}$$

The conversion matrix is obviously triangular, and can be shown (after judicious use of properties of the binomial coefficients) to be its own inverse. The identity to be exploited is

$$\sum_k (-1)^k \binom{j}{k} \binom{k}{l} = \delta_{j,l}$$

where it is sufficient to take the sum over all k for which the upper index of the binomial coefficients is greater than or equal to the lower index.
So in the general case

$$A_i = \sum_k (-1)^i \binom{k}{i} a_k$$

and hence for any given ARMA filter with coefficients a_k

$$y_n = a_0 x_n + a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_L x_{n-L}$$

we can always find an equivalent difference equation

$$y_n = A_0 x_n + A_1 \Delta x_n + A_2 \Delta^2 x_n + \ldots + A_L \Delta^L 3 x_n$$

and vice versa, and the coefficients are related by the relations found above.