6.3.2 We saw that a generalized echo system $\mathbf{y}(t) = x(t) + hx(t-\tau)$ has no zeros in its frequency response for $|h| < 1$; i.e., there are no sinusoids that are exactly canceled out. Are there signals that are canceled out by this system?

Yes there are, although they will not be sinusoids. For example, the signal given by $x(t) = Ae^{-\lambda t}$ where we pick λ such that $h = -e^{-\lambda\tau}$ gives

$$\begin{aligned} y(t) &= x(t) + hx(t-\tau) \\ &= Ae^{-\lambda t} + hAe^{-\lambda(t-\tau)} \\ &= Ae^{-\lambda t}\left(1 + he^{\lambda\tau}\right) \\ &= 0 \end{aligned}$$

is seen to zero the system output. This can be made mode complex, e.g. an exponentially decaying or increasing sinusoid.

Even more simple is the digital case. We are looking for a digital signal x_n for which $y_n = x_n + hx_{n-1}$ is identically zero. The signal $x_n = A(-h)^n$ is obviously such a signal.

$$\begin{aligned} y_n &= x_n + hx_{n-1} \\ &= A(-h)^n + hA(-h)^{n-1} \\ &= A(-h)^n\left(1 + \frac{h}{-h}\right) \\ &= 0 \end{aligned}$$

The problem is that there seems to be an argument that proves that the only signal x_n that can give identically zero output is the zero signal. The 'proof' goes as follows. Let's assume that there is such as signal x which is *not* identically zero; then at least one of its Fourier components must be nonzero. Let's call this frequency Ω, so that $X(\Omega) \neq 0$.

Now we know that if $|h| < 1$ the frequency response has no zeros, i.e. $H(\omega) \neq 0$ for all ω, and specifically for Ω we can say that $H(\Omega) \neq 0$.

Let's look at the output signal in the frequency domain at frequency Ω.

$$Y(\Omega) = H(\Omega)X(\Omega) \neq 0$$

However, if there is a frequency at which $Y(\Omega) \neq 0$ then $y(t)$ can not be the zero signal, thus contradicting the premise that $x(t)$ causes $y(t)$ to be identically zero.

The problem is the implicit assumption that $x(t)$ *has* a Fourier transform. All the examples we saw above, and indeed all signals that are cancelled out by systems with $h \neq 1$, are exponentially increasing or decaying. For example, $x(t) = Ae^{-\lambda t}$ becomes infinite as we go back in time, and so its Fourier integral diverges. The solution is to use the Laplace or z transforms. Then $Y(z) = H(z)X(z)$ can equal zero for a nonzero $X(z)$ since $H(z)$ has a zero somewhere on the real axis inside the unit circle.

A related way of solving the problem is to change the solution a little bit. For example, the signal

$$x(t) = \begin{cases} 0 & t < 0 \\ Ae^{-\lambda t} & t \geq 0 \end{cases}$$

does have a FT, and does cause $y(t) = 0$ most of the time. Yet $y(t)$ displays a spike within τ of $t = 0$ and so is not identically zero.