Finding poles and zeros of a filter.

Let's start with a filter in the usual (a_l, b_m) form.

$$y_n = x_n - \frac{3}{2}x_{n-1} + \frac{1}{2}x_{n-2} - y_{n-1} - \frac{1}{2}y_{n-2}$$

First, we create the symmetric (α_l, β_m) form by moving all the y terms to the left side.

$$y_n + y_{n-1} + \frac{1}{2}y_{n-2} = x_n - \frac{3}{2}x_{n-1} + \frac{1}{2}x_{n-2}$$

Next, we write this as an equation for signals (rather than an equation for values in the time domain).

$$\left(1 + \hat{z}^{-1} + \frac{1}{2}\hat{z}^{-2}\right) y = \left(1 - \frac{3}{2}\hat{z}^{-1} + \frac{1}{2}\hat{z}^{-2}\right) x$$

Now we take the z transform of both sides, using the fundamental theorem $\mathrm{zT}(\hat{z}^{-1}x) = z^{-1}\mathrm{zT}(x)$.

$$\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right) Y(z) = \left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right) X(z)$$

This means that

$$Y(z) = \frac{\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)}{\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)} X(z)200$$

But $Y(z) = H(z)X(z)$ so we have found the *transfer function* of this filter:

$$H(z) = \frac{\left(1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}\right)}{\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Multiplying top and bottom by z^2 we obtain

$$H(z) = \frac{\left(z^2 - \frac{3}{2}z + \frac{1}{2}\right)}{\left(z^2 + z + \frac{1}{2}\right)}$$

which can be factored as follows:

$$H(z) = \frac{(z-1)(z-\frac{1}{2})}{(z+\frac{1}{2}(1+\mathrm{i}))(z+\frac{1}{2}(1-\mathrm{i}))}$$

which is a rational function (the ratio of two polynomials in z).

The zeros of the transfer function are the roots of the polynomial in the numerator. These are easily seen to be 1 and $\frac{1}{2}$.

The poles of the transfer function are the roots of the polynomial in the denominator. A little algebra shows that these are $-\frac{1}{2}(1 \pm \mathrm{i})$.

We see that there are zeros to the left of the y axis (low frequencies), including on at DC, and there are poles to the right of the y axis (high frequencies), so we can conclude that this is a high-pass filter. To understand this, you can input DC ($x_n = \ldots{+1}{+1}{+1}{+1}\ldots$) and Nyquist ($x_n = \ldots{-1}{+1}{-1}{+1}\ldots$) to the original equation in the time domain and see what you get. Alternatively, look at the transfer function only on the unit circle by substituting $z = e^{\mathrm{i}\omega n}$ and find the frequency response $H(\omega)$.

Finally, we can draw the *pole-zero diagram* of the filter, which determines the filter to within a gain factor.