9.11 Line Spectral Pairs

Another set of parameters that contain exactly the same information as the LPC coefficients are the **L**ine **S**pectral **P**air (LSP) frequencies. To introduce them we need to learn a mathematical trick that can be performed on the polynomial in the denominator of the LPC system function.

A polynomial of degree M

$$a(x) = \sum_{m=0}^{M} p_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots a_{M-2} x^{M-2} + a_{M-1} x^{M-1} + a_M x^M$$

is called 'palindromic' if $a_m = a_{M-m}$, i.e.,

$$a_0 = a_M \qquad a_1 = a_{M-1} \qquad a_2 = a_{M-2} \qquad \text{etc.}$$

and 'antipalindromic' if $a_m = -a_{M-m}$, i.e.,

$$a_0 = -a_M \qquad a_1 = -a_{M-1} \qquad a_2 = -a_{M-2} \qquad \text{etc.}$$

so $1 + 2x + x^2$ is palindromic, while $x + x^2 - x^3$ is antipalindromic. It is not hard to show that the product of two palindromic or two antipalindromic polynomials is palindromic, while the product of an antipalindromic polynomial with a palindromic one is antipalindromic.

We will now prove that every real polynomial that has all of its zeros on the unit circle is either palindromic or antipalindromic. The simplest cases are $x+1$ and $x-1$, which are obviously palindromic and antipalindromic, respectively. Next consider a second degree polynomial with a pair of complex conjugate zeros on the unit circle.

$$\begin{aligned} a(x) &= \left(x - e^{\mathrm{i}\phi}\right)\left(x - e^{-\mathrm{i}\phi}\right) \\ &= x^2 - e^{-\mathrm{i}\phi} x - e^{\mathrm{i}\phi} x + e^{\mathrm{i}\phi} e^{-\mathrm{i}\phi} \\ &= x^2 - 2\cos(\phi) + 1 \end{aligned}$$

This is obviously palindromic.

Any real polynomial that has k pairs of complex conjugate zeros will be the product of k palindromic polynomials, and thus palindromic. If a polynomial has k pairs of complex conjugate zeros and the root $+1$ it will also be palindromic, while if it has -1 as a root it will be antipalindromic. This completes the proof.

The converse of this statement is not necessarily true; not every palindromic polynomial has all its zeros on the unit circle. The idea behind the

LSPs is to define palindromic and antipalindromic polynomials that *do* obey the converse rule. Let's see how this is done.

Any arbitrary polynomial $a(x)$ can be written as the sum of a palindromic polynomial $p(x)$ and an antipalindromic polynomial $q(x)$

$$a_m = \frac{1}{2}(p_m + q_m) \qquad \text{where} \qquad \begin{array}{rcl} p_m &=& a_m + a_{M-m} \\ q_m &=& a_m - a_{M-m} \end{array} \tag{9.39}$$

(if M is even the middle coefficient appears in p_m only). When we are dealing with polynomials that have their constant term equal to unity, we would like the polynomials p_m and q_m to share this property. To accomplish this we need only pretend for a moment that a_m is a polynomial of order $M+1$ and use the above equation with $a_{M+1} = 0$.

$$a_m = \frac{1}{2}(p_m + q_m) \qquad \text{where} \qquad \begin{array}{rcl} p_m &=& a_m + a_{M+1-m} \\ q_m &=& a_m - a_{M+1-m} \end{array} \tag{9.40}$$

Now $a_0 = p_0 = q_0 = 1$ but p_m and q_m are polynomials of degree $M+1$.

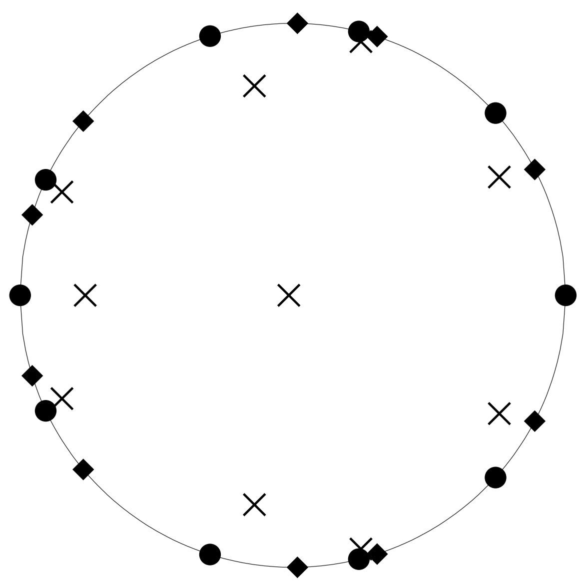

Figure 9.5: The zeros of a polynomial and of its palindromic and antipalindromic components. The **X**s are the zeros of a randomly chosen tenth order polynomial (constrained to have its zeros inside the unit circle). The circles and diamonds are the zeros of the $p(x)$ and $q(x)$. Note that they are all on the unit circle and are intertwined.

Formally we can write the relationships between the polynomials

$$a(x) = \frac{1}{2}\Big(p(x) + q(x)\Big) \qquad \text{where} \qquad \begin{pmatrix} p(x) \\ q(x) \end{pmatrix} = a(x) \pm x^{M+1} a(x^{-1})$$

and it is not hard to show that if all the zeros of $a(x)$ are inside the unit circle, then all the zeros of $p(x)$ and of $q(x)$ are *on* the unit circle. Furthermore, the zeros of $p(x)$ and $q(x)$ are intertwined, i.e., between every two zeros of $p(x)$ there is a zero of $q(x)$ and vice versa. Since these zeros are on the unit circle they are uniquely specified by their angles. For the polynomial in the denominator of the LPC frequency response these angles represent frequencies, and are called the LSP frequencies.

Why are the LSP frequencies a useful representation of the all-pole filter? The LPC coefficients are not a very homogeneous set, the higher-order b_m being more sensitive than the lower-order ones. LPC coefficients do not quantize well; small quantization error may lead to large spectral distortion. Also the LPC coefficients do not interpolate well; we can't compute them at two distinct times and expect to accurately predict them in between. The zeros of the LPC polynomial are a better choice, since they all have the same physical interpretation. However, finding these zeros numerically entails a complex two-dimensional search, while the zeros of $p(x)$ and $q(x)$ can be found by simple one-dimensional search techniques. In speech applications it has been found empirically that the LSP frequencies quantize well and interpolate better than all other parameters that have been tried.

EXERCISES

9.11.1 Let's create a random polynomial of degree M by generating $M+1$ random numbers and using them as coefficients. We can now find the zeros of this polynomial and plot them in the complex plane. Verify empirically the hard-to-believe fact that for large M most of the zeros are close to the unit circle (except for large negative real zeros). Change the distribution of the random number generator. Did anything change? Can you explain why?

9.11.2 Prove that if all the zeros of $a(x)$ are inside the unit circle, then all the zeros of $p(x)$ and of $q(x)$ are *on* the unit circle. (Hint: One way is write the p and q polynomials as $a(x)\,(1 \pm h(x))$ where $h(x)$ is an all-pass filter.) Prove that the zeros of $p(x)$ and $q(x)$ are intertwined. (Hint: Show that the phase of all-pass filter is monotonic, and alternately becomes π (zero of p) and 0 (zero of q).)

9.11.3 A pipe consisting of $M+1$ cylinders that is completely open or completely closed at the end has its last reflection coefficient $k_{M+1} = \pm 1$. How does this relate to the LSP representation?

9.11.4 Generate random polynomials and find their zeros. Now build $p(x)$ and $q(x)$ and find their zeros. Verify that if the polynomial zeros are inside the unit circle, then those of p and q are on the unit circle. Is there a connection between the angles of the polynomial zeros and those of the LSPs?

9.11.5 The Greek mathematician Apollonius of Perga discovered that given two points in the plane z_1 and z_2, the locus of points with distances to z_1 and z_2 in a fixed ratio is circle (except when the ratio is fixed at one when it is a straight line). Prove this theorem. What is the connection to