Part 1
Signals

0368.3464

עיבוד ספרתי של אותות

Digital Signal Processing for Computer Science

AKA

Digital Signal Processing – Algorithms and Applications

WARNING: This is a very different course from DSP for Engineering students

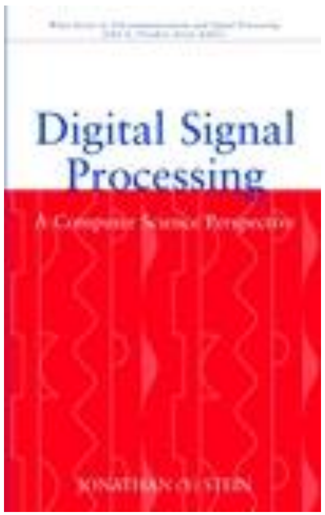
The course

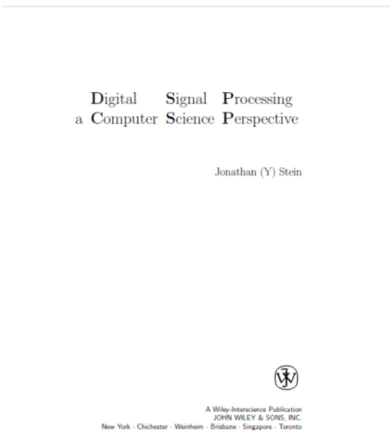
- Always check the Moodle
- We’ll start at 17:10
- One break 18:30-1845, finish at 19:45

Requirements

- **Lecture attendance** is mandatory
  - attendance checked from the third lecture on
  - active participation is better
  - if you can’t make it now and again – there will be recordings
  - 2/3 participation is required to take the final exam
- **No homework assignments**
- Problems? use the Moodle forum

The course text




Digital Signal Processing
a Computer Science Perspective

Jonathan (Y) Stein

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עיבוד ספרתי של אותות - מבט מדעי המחשב

WWW.DSPCSP.COM

Why did I write this book?

The book started in a course I gave back in 1996

Until DSPCSP only engineering students learned DSP
and DSP is mostly programming!

Engineering students understood DSP
but not algorithms and not how to program
(one well-known DSP book by an Israeli author
apologizes for introducing the FFT algorithm!)

Computer Science students knew algorithms and programming
but the existing books were incomprehensible to them
(and its not just the strange usage of $j = \sqrt{-1}$
it is the whole mindset)

This book and course bridges the gap!
and there are now many more books like it (but not as good ☺)

A few more things

- The course will focus on *understanding* the main concepts
  not mathematical formulas or formal proofs of theorems
- Hopefully, you will come away with an understanding
  of how many many things work nowadays
  and will remember these for many years
- If there are applications that interest you

  e.g., radar, predicting stock market trends, musical effects, …

  we can talk about some of them in the course

Course Outline

1. Signals (analog and digital)
2. Spectrum (frequency domain)
3. Sampling
4. Transforms
5. Systems
6. Filters
7. Convolution
8. MA, AR, ARMA filters
9. System identification
10. Graph theory
11. FFT
12. DSP processors
13. Correlation and Adaptation
14. Speech signal processing
15. Data communications
16. …

PART 1 SIGNALS (1–4)

PART 2 SYSTEMS (5–9)

PART 3 ALGORITHMS (10–12)

PART 4 APPLICATIONS (13–16)

Questions?

Any questions about logistics?

(Please don't ask about the final exam yet …)

???

Motivation

What is DSP good for?

You are using
Digital **S**ignal **P**rocessing
right now
and are probably carrying a few
Digital **S**ignal **P**rocessors
with you right now!

What is this?

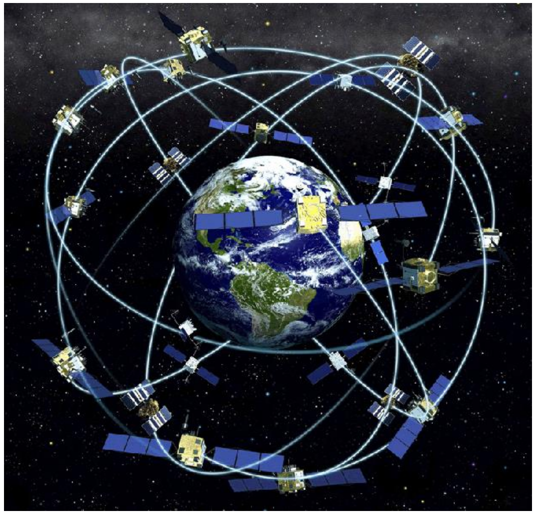
BOSS RC-505 Loop Station

Let's see it in (in)action

Let's see someone who really knows about DSP

Some music effects are *easy to understand*

How does GPS work?



[GPS and the quest for Pizza]

But what signals do the satellites transmit?

and how does the GPS receiver know

WHEN a signal was received?

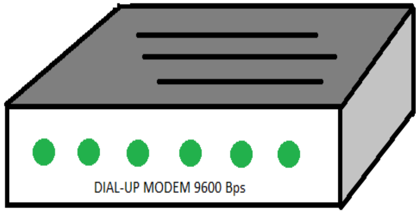
Telephony

How many of you have seen one of these?

How many of you still have one of these?

How many of you have used one of these?

One of these?



Wonderful tones!

- dial and busy
- dial and ring
- different DTMF tones
- T.30 fax
- V.34 modem
- different answer tones
- musical instruments

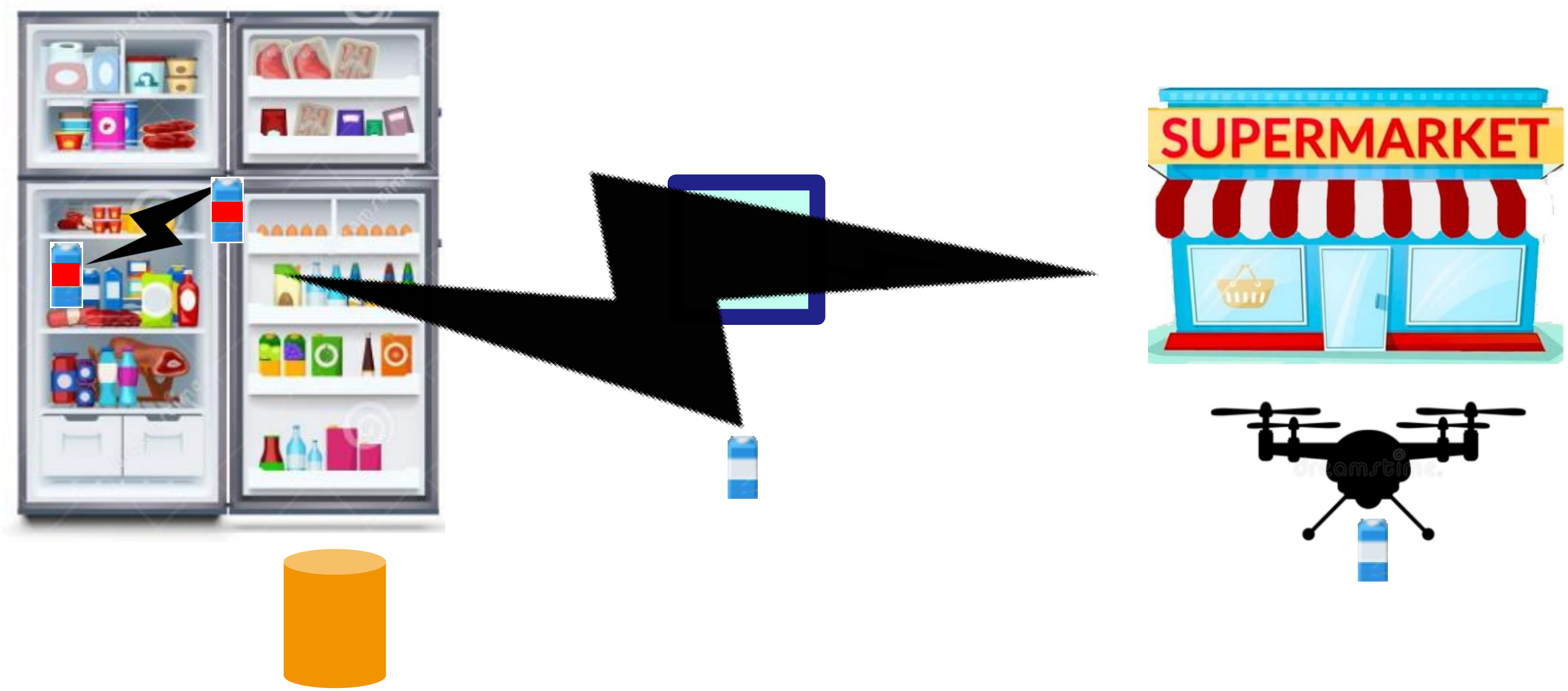
5G

We have all heard that 5G is the NEXT THING

It is so much better than 4G …

How does it do it?

The 5G refrigerator



Speech

- Speech synthesis – text to speech (demo)
- Speech recognition – speech to text (demo)
- Speaker recognition
- Speaker verification
- Speech compression
- Dynamic Time Warping
- Language recognition
- Speech polygraph

Deep fakes

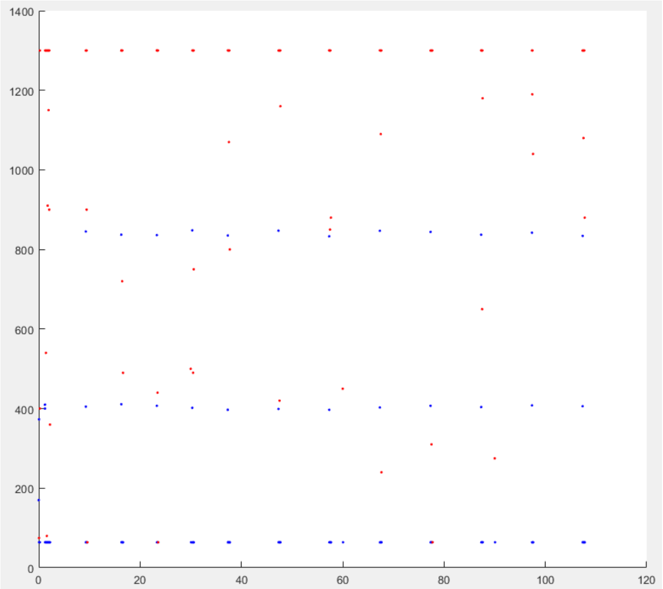
I'm sure you have all heard this

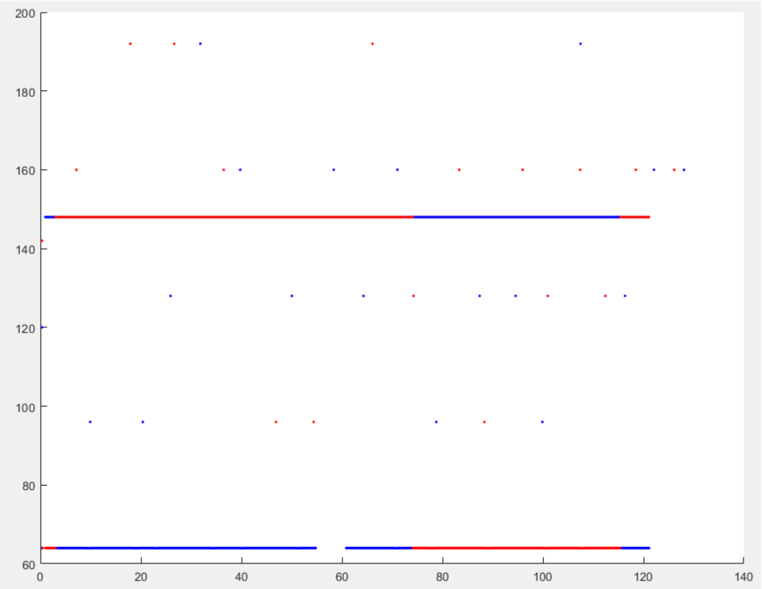
Classifying Encrypted Traffic

We all want our Internet traffic to be private

but we also want good **Q**uality **o**f **E**xperience

How can an Internet Service Provider do this?





Let’s start - What is DSP?

Digital **S**ignal **P**rocessing

Digital (Signal Processing)

עיבוד ספרתי של אותות

(Digital Signal) Processing

עיבוד של אותות ספרתיים

What is a signal?

There is no such thing as a signal !

But there is an **analog signal** and a **digital signal**

An analog signal s(t) is

a real function of a single variable called time (t)

But not just *any* function – in a moment we will see conditions

A digital signal s_n is

a real sequence with a single index called (discrete) time (n)

But not just *any* sequence – in a moment we will see conditions

And there is a connection between analog and digital signals !

What isn't a signal? (Part 1)

- Complex functions z(t) or sequences z_n
  (they aren't real!)
- Images I(x,y)
  (it is two dimensional – not a scalar function/sequence)
- Videos v(x,y,t)
  (it is three dimensional)
- Waves w(**x**,t)
  (they are functions of *space* and time, not just of time)
- Information (*I hope you know what that means* …)
  (but signals can *carry* information)

DSP

Digital Signal Processing vs. **Analog** Signal Processing

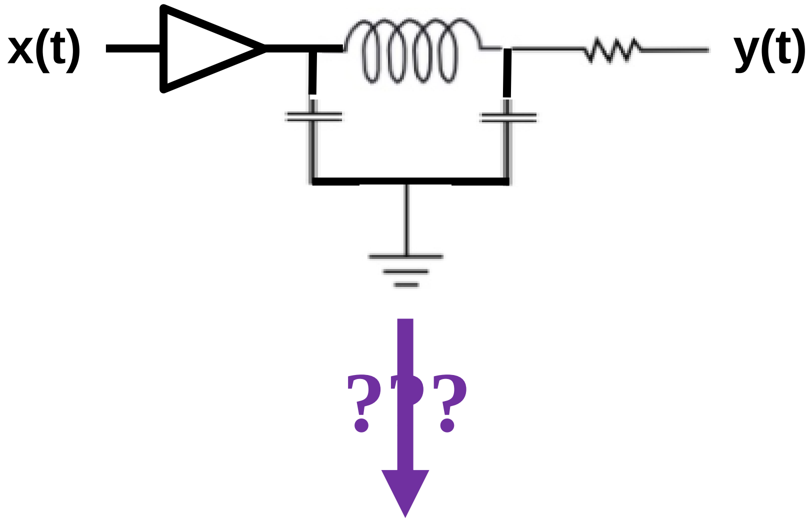
Why **D**SP ? use (digital) computer instead of (analog) electronics

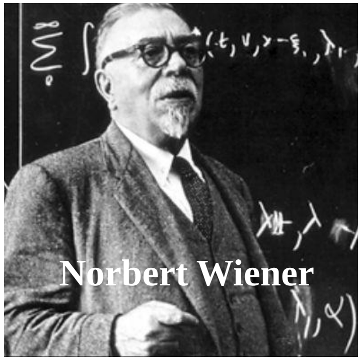
- more flexible
  - new functionality requires code changes, not component changes
- more accurate
  - even simple amplification can not be done exactly in electronics
- more stable
  - code performs consistently
- more sophisticated
  - can perform more complex algorithms (e.g., SW receiver)

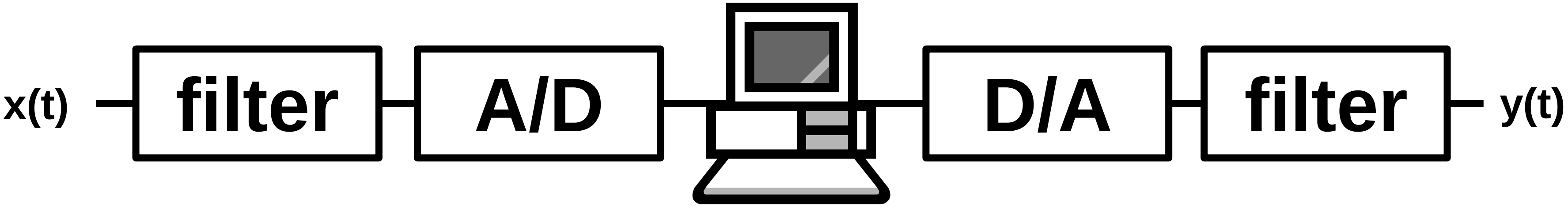
However

- digital computers only process sequences of numbers
  - not analog signals
- requires converting analog signals to digital domain for processing
- and digital signals back to analog domain

Is it worthwhile?







Yes, because of

- feasibility (there are operations that are impossible in analog)
- flexibility (it is very difficult to upgrade analog hardware)
- accuracy (even simple *gain* is not completely accurate in analog)

Signals

Analog signal

$s(t)$

continuous time

$-\infty < t < +\infty$

Digital signal

s_n

discrete time

$n = -\infty \ldots +\infty$

(unlike sequences in math)

Physicality requirements

- s values are real
- s values defined for all times
- Finite energy
- Finite bandwidth

Mathematical usage

- s may be complex
- s may be singular
- Infinite energy allowed
- Infinite bandwidth allowed

Energy = how "big" the signal is

Bandwidth = how "fast" the signal is

(we'll see the exact definitions later)

Show me the money!

Why are finite energy and bandwidth *physicality requirements*?

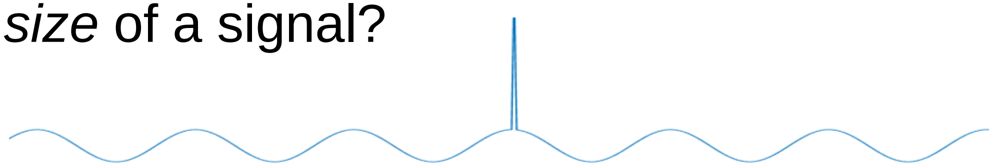
Energy of a signal is related to energy in physics
energy is conserved, so we are willing to pay for it!
(electric bill, gas for car, food)
A signal with infinite energy
would cost infinite money to generate!

Bandwidth of a signal
is related to how much information the signal carries
information always decreases (entropy always increases)
so we are willing to pay for it!
(Internet, cellular, books, newspapers)
A signal with infinite bandwidth
would cost infinite money to generate!

Energy

How can we capture the *size* of a signal?

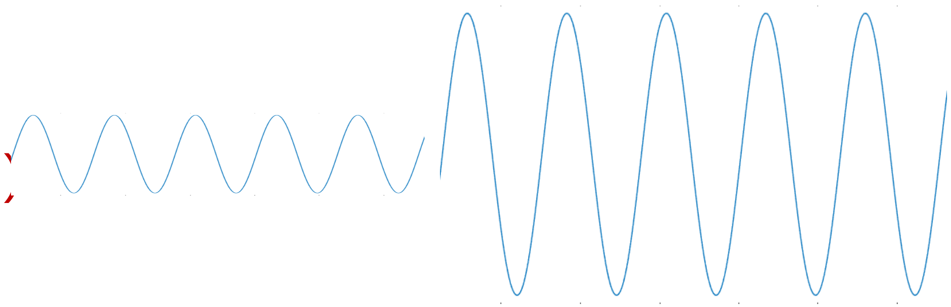
the maximum value?



the average value ?

both of these are zero!

(this is the DC component!)

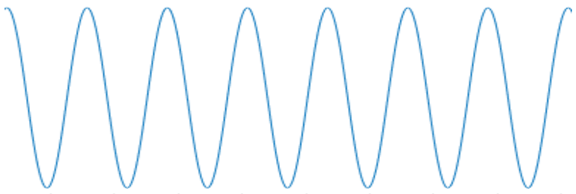


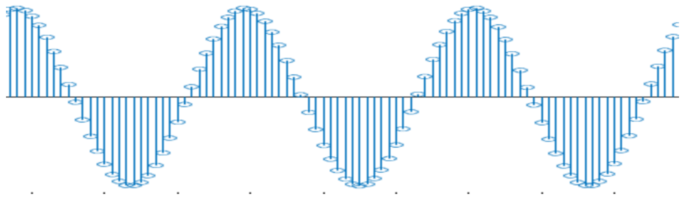
The natural definition is

$$E = \int_{-\infty}^{\infty} |s(t)|^2 \ dt \qquad\qquad E = \sum_{n=-\infty}^{\infty} |s_n|^2$$

Can you think of other *good* definitions?

Handling infinite energy





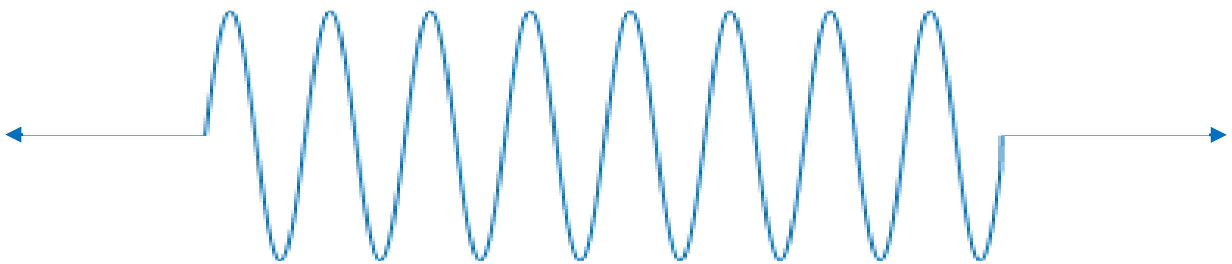
Sinusoids

are among the most important *signals* we will use

But are infinite in extent and thus have infinite energy

So we *fix* them (limit them to a finite time duration)

by multiplying them by a *gating signal*



illegal signals

DSP is not just mathematics
it is a technology for interaction with the real world

The conditions guarantee that signals are real objects
of the kind we find in the real world

However, requiring every object to conform to the restrictions
would make the mathematics very hard

So we will often use objects which do not obey the conditions

- complex functions/sequences
- objects with infinite energy

and even call them signals!

But this is just to simplify the math
the answers will be the same!

What isn't a signal? Part 2

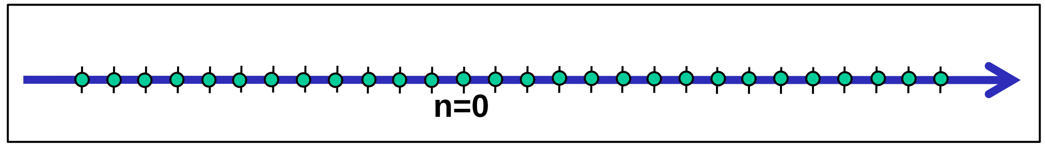
2.1.1 Which of the following are *signals*? Explain which requirement of the definition is possibly violated and why it is acceptable or unacceptable to do so.

1. the height of Mount Everest
2. $(e^{it} + e^{-it})$
3. the price of a slice of pizza
4. the 'sinc' function $\frac{\sin(t)}{t}$
5. Euler's totient function $\phi(n)$, the number of positive integers less than n having no proper divisors in common with n
6. the water level in a toilet's holding tank
7. $\lfloor t \rfloor$ the greatest integer not exceeding t
8. the position of the tip of a mosquito's wing
9. $\sqrt{t}$
10. the Dow Jones Industrial Average
11. $\sin(\frac{1}{t})$
12. the size of water drops from a leaky faucet
13. the sequence of values x_n in the interval $[0 \ldots 1]$ defined by the *logistics recursion* $x_{n+1} = \lambda x_n (1 - x_n)$ for $0 \leq \lambda \leq 4$

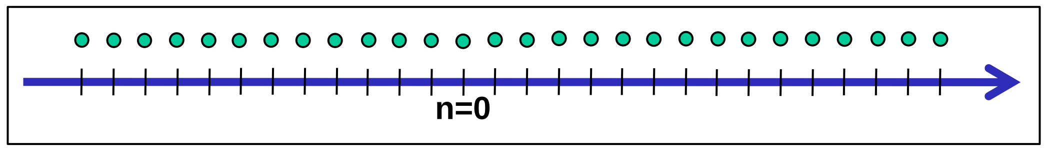
After trying look here - http://www.dspcsp.com/exercises/X2-1-1.pdf

Some digital signals

Zero signal
$s_n = 0$

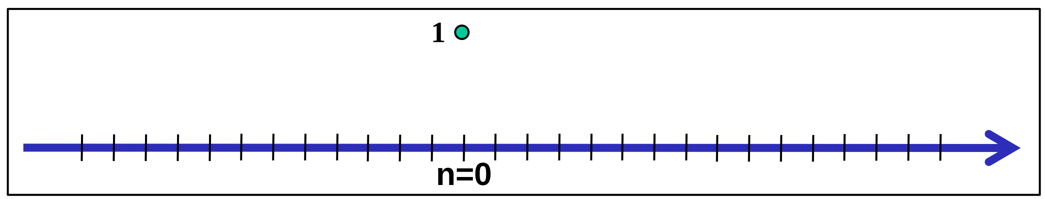


Constant signal
(∞ energy!) $s_n = k$

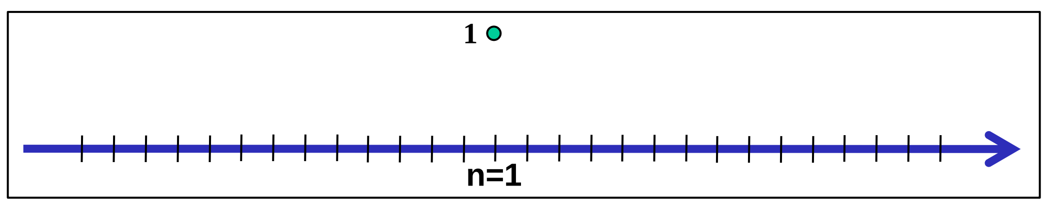


dot plot

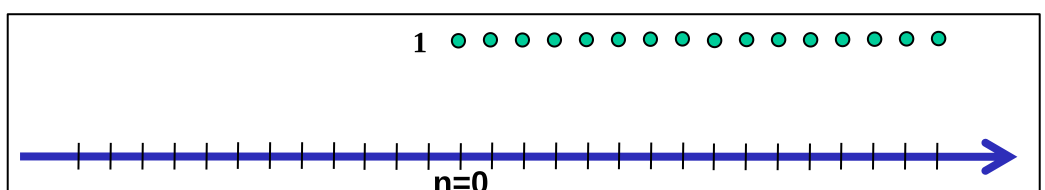
Unit Impulse (UI)
$s_n = \delta_{n,0}$



Shifted Unit Impulse (SUI)
$s_n = \delta_{n,m}$



Step (∞ energy!)
$s_n = \theta_n$



Signals as *objects*

Signals are **more** than just a collection of values

(we call the collection of values the signal's representation)

For example, we can perform operations on signals

- gain (and attenuation) $y = g\,x$ g is a *number*, x and y are *signals*

  means $\forall n = -\infty \ldots +\infty \quad y_n = g\,x_n$ $\quad \forall t \;\; -\infty \le t \le +\infty \quad y(t) = g\,x(t)$

  special cases $g = -1$ (inversion), $g<0$

- add 2 signals $w = x + y$ x and y are *signals*

  means $\forall n = -\infty \ldots +\infty \quad w_n = x_n + y_n$ $\quad w(t) = x(t) + y(t)$

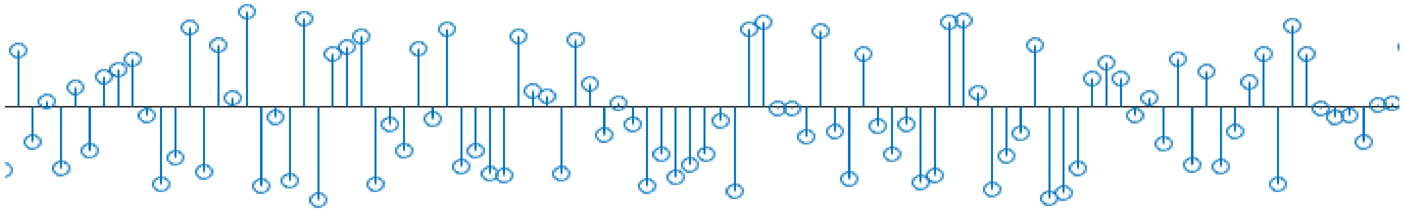
- what does $w = x - y$ mean?
- what does $w = ax + by$ mean? (a,b *numbers*, w,x,y *signals)*

Deterministic vs stochastic signals

Signals (analog or digital) can be *deterministic* or *stochastic*

- *deterministic* means that there is some algorithm
  that enables us to predict the signal for all time
- *Stochastic means nondeterministic*
  the signal is *random* in some sense

The most stochastic signal of them all is *white noise*

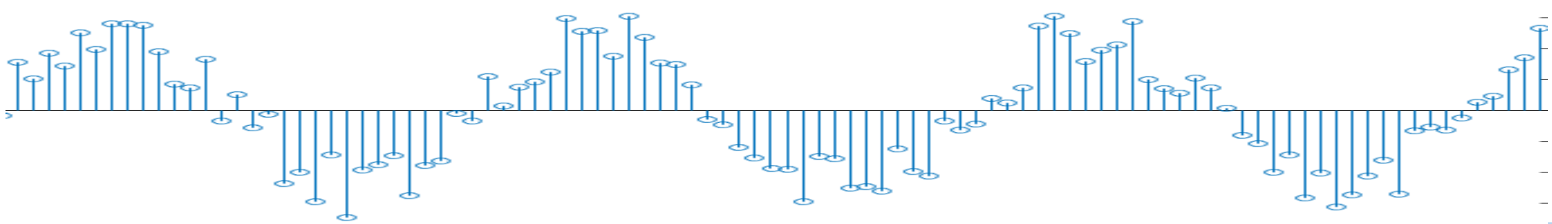


stem plot

even if we observe white noise w_n from $-\infty$ to n

we can't say anything about w_{n+1}

example of non-white stochastic signal



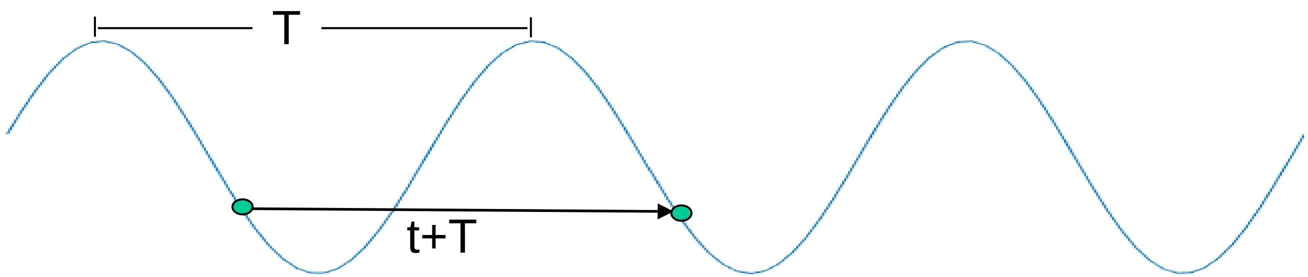
Periodic signals

Signals (analog or digital) can be periodic

- Analog periodic signal p(t)

  $\exists$ T > 0 s.t. $\forall$t $-\infty \le t \le +\infty$ $p(t+T) = p(t)$

  the smallest such T is called the *period*



- Digital periodic signal p_n

  $\exists$ N > 0 s.t. $\forall$n= $-\infty \ldots +\infty$ $p_{n+N} = p_n$

  the smallest such N is called the *period*

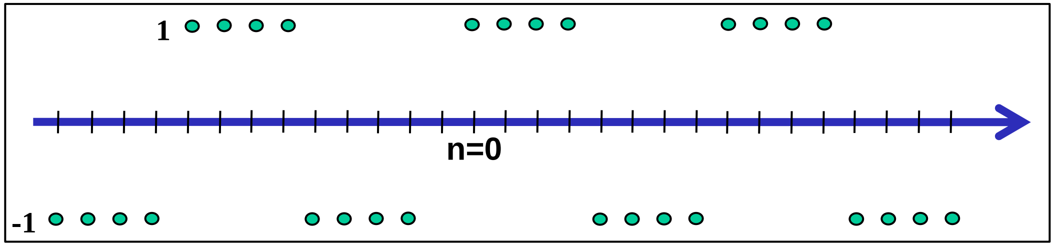
  Is the digital sinusoid $s_n = A \sin(\omega n)$ always periodic? If not, when is it?

Only deterministic signals can be *periodic*

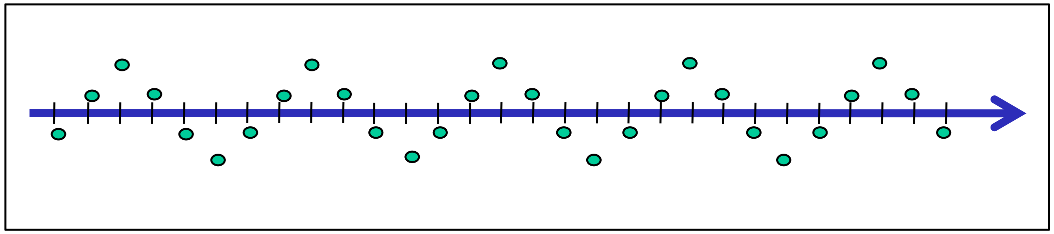
why can't a periodic signal be stochastic?

Some periodic digital “signals”

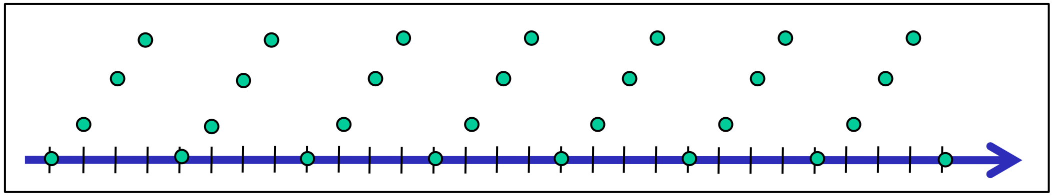
Square wave



Triangle wave

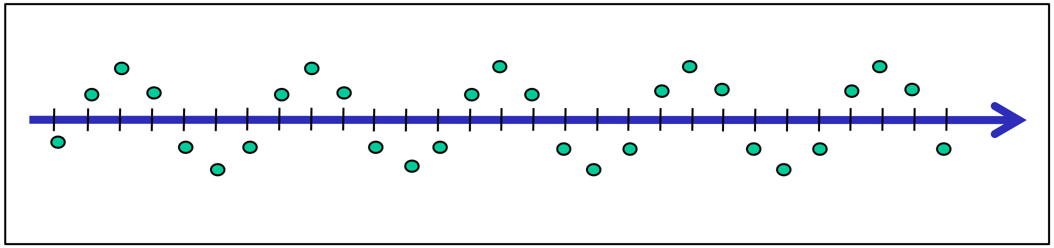


Saw tooth

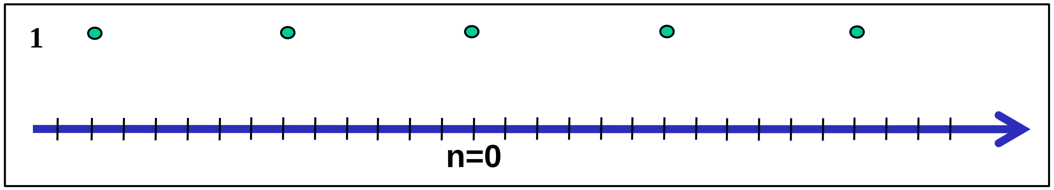


Sinusoid

(not always periodic!)



Pulse train



The $\hat{z}$ operator

hat means operator – not gain!

we'll see later why it is called "z"

For ***digital*** signals we define

The time *advance* operator

$$y = \hat{z}\, x \quad \text{means} \quad \forall n = -\infty \ldots +\infty \quad y_n = x_{n+1}$$

This operator is noncausal (needs a crystal ball)

for what kind of signal can we always implement?

The time *delay* operator

$$y = \hat{z}^{-1} x \quad \text{means} \quad \forall n = -\infty \ldots +\infty \quad y_n = x_{n-1}$$

This operator is causal (can always be implemented)

What do you think $y = \hat{z}^m x$ means?

What is $\hat{z}^{-1}\hat{z}$? $\hat{z}\hat{z}^{-1}$?

Why aren't $\hat{z}$ and $\hat{z}^{-1}$ defined for analog signals?

Some more operations

- first finite difference $y = \hat{\Delta} x$ means $y_n = x_n - x_{n-1}$
  - note: $\hat{\Delta} = 1 - \hat{z}^{-1}$

and there are higher order finite differences $y = \hat{\Delta}^m x$

n		-2	-1	0	1	2	…
x	…	x_{-2}	x_{-1}	x_0	x_1	x_2	…
$\hat{\Delta} x$	…	$x_{-2} - x_{-3}$	$x_{-1} - x_{-2}$	$x_0 - x_{-1}$	$x_1 - x_0$	$x_2 - x_1$	…
$\hat{\Delta}^2 x$	…	…	…	$(x_0 - x_{-1}) - (x_{-1} - x_{-2}) =$ $x_0 - 2x_{-1} + x_{-2}$	…	…	…

If the signal is a polynomial in time n what can we say about $\hat{\Delta}^m x$?

Some more operations

Here are some more operations

- the accumulator y = $\hat{\gamma}$ x is a *running* summation

  $$y_n = \sum_{m=-\infty}^{n} x_m \quad = \quad y_{n-1} + x_n$$

  the accumulator is the inverse of the finite difference

  $$\hat{\gamma}\,\hat{\Delta} = \hat{\Delta}\;\hat{\gamma} = \hat{1}$$

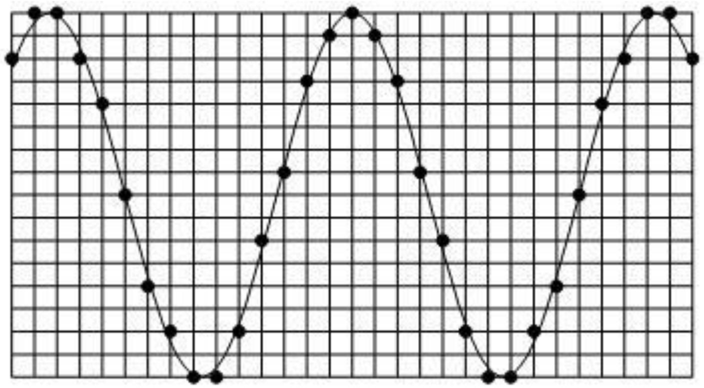
- time reversal : y = Rev(x) means $y_{\mathbf{n}} = x_{\mathbf{-n}}$
- we can compare two signals - how similar are they?

  $$Cxy(m) = \sum_{n=-\infty}^{\infty} x_n\; y_{n-m}$$

- the Hilbert transform $\hat{H}$ (see later)

Sampling

From an analog signal we can create a digital signal
by **SAMPLING**



Under certain conditions
we can uniquely return to the analog signal !

Even though the digital signal has only $\aleph_0$ values
and the analog has $\aleph_1$

This sounds impossible!

How can we know what happens between 2 samples?

Digital signals and vectors

Digital signals are in many ways like ***vectors***

$$\ldots\ s_{-5}\ s_{-4}\ s_{-3}\ s_{-2}\ s_{-1}\ s_0\ s_1\ s_2\ s_3\ s_4\ s_5\ \ldots \quad \leftrightarrow \quad (x, y, z)$$

In fact

- the zero vector 0 ($0_n = 0$ for all times n)
- every two signals can be added to form a new signal x + y = z
- every signal can be multiplied by a real number (amplified!)
- every signal has an inverse signal -s so that s + -s = 0 (zero signal)
- every signal has a length - its energy

So, they form a **linear vector space** (with norm)

Similarly, analog signals, periodic signals with given period, etc.

all form linear vector spaces

Time

However, signals are not *only* vectors

With regular vectors

the ordering of the components is arbitrary

We can decide to list them (x,y,z) or (y,z,x) or (z,y,x) !

For digital signals the component order is **not** arbitrary

since time is ordered and flows in one direction !

That's why we could define

- time advance operator $\hat{\mathbf{z}}$ $\quad (\mathbf{z}\ \mathbf{s})_n = \mathbf{s}_{n+1}$
- time delay operator $\hat{\mathbf{z}}^{-1}$ $\quad (\mathbf{z}^{-1}\ \mathbf{s})_n = \mathbf{s}_{n-1}$

these wouldn't make sense for vectors

This is the arrow of entropic time

Bases

the **fundamental theorem in linear algebra**

All linear vector spaces have a basis (usually > 1 !)

A basis is a set of vectors $\mathbf{b_1}\ \mathbf{b_2} \dots \mathbf{b_d}$ that obeys 2 conditions :

1. spans the vector space

   i.e., for every vector **x** : $\mathbf{x} = a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \dots + a_d \mathbf{b_d}$

   where $a_1 \dots a_d$ are a set of coefficients

2A the basis vectors $\mathbf{b_1}\ \mathbf{b_2} \dots \mathbf{b_d}$ are linearly independent

   i.e., *if* $a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \dots + a_d \mathbf{b_d} = \mathbf{0}$ (the zero vector)

   then $a_1 = a_2 = \dots = a_d = 0$

OR

2B The expansion $\mathbf{x} = a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \dots + a_d \mathbf{b_d}$ is unique

(we'll prove that these 2 statements are equivalent)

Since the expansion is unique

   the coefficients $a_1 \dots a_d$ *represent* the vector in that basis

Equivalence

1. A → B

Given: the basis is linearly independent

$a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \ldots + a_d \mathbf{b_d} = \mathbf{0} \;\; \rightarrow a_1 = a_2 = \ldots = a_d = 0$

Assume that the representation is *not unique*

$\mathbf{x} = a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \ldots + a_d \mathbf{b_d} \;\; \mathbf{=} c_1 \mathbf{b_1} + c_2 \mathbf{b_2} + \ldots + c_d \mathbf{b_d}$

By the definition of zero and that of subtraction of 2 vectors

$0 = x\text{-}x = (a_1 - c_1)\, \mathbf{b_1} + (a_2 - c_2)\, \mathbf{b_2} + \ldots + (a_d - c_d)\, \mathbf{b_d}$

From the assumption $a_1 = c_1 \;\; a_2 = c_2 \;\; \ldots \;\; a_d = c_d$

2. B → A

Given: the representation is unique

$\mathbf{x} = a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \ldots + a_d \mathbf{b_d}$

Assume that the basis is **linearly dependent**

$a_1 \mathbf{b_1} + a_2 \mathbf{b_2} + \ldots + a_d \mathbf{b_d} = \mathbf{0}$ and $a_1 \neq 0$ and/or … $a_d \neq 0$

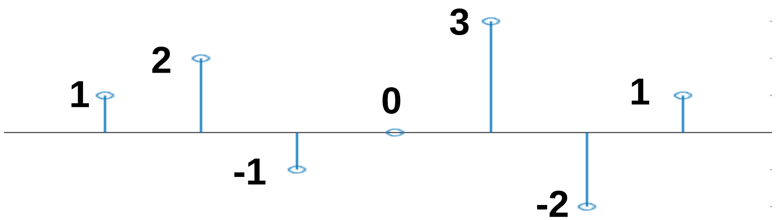
Then the representation of the vector **0** is not unique!

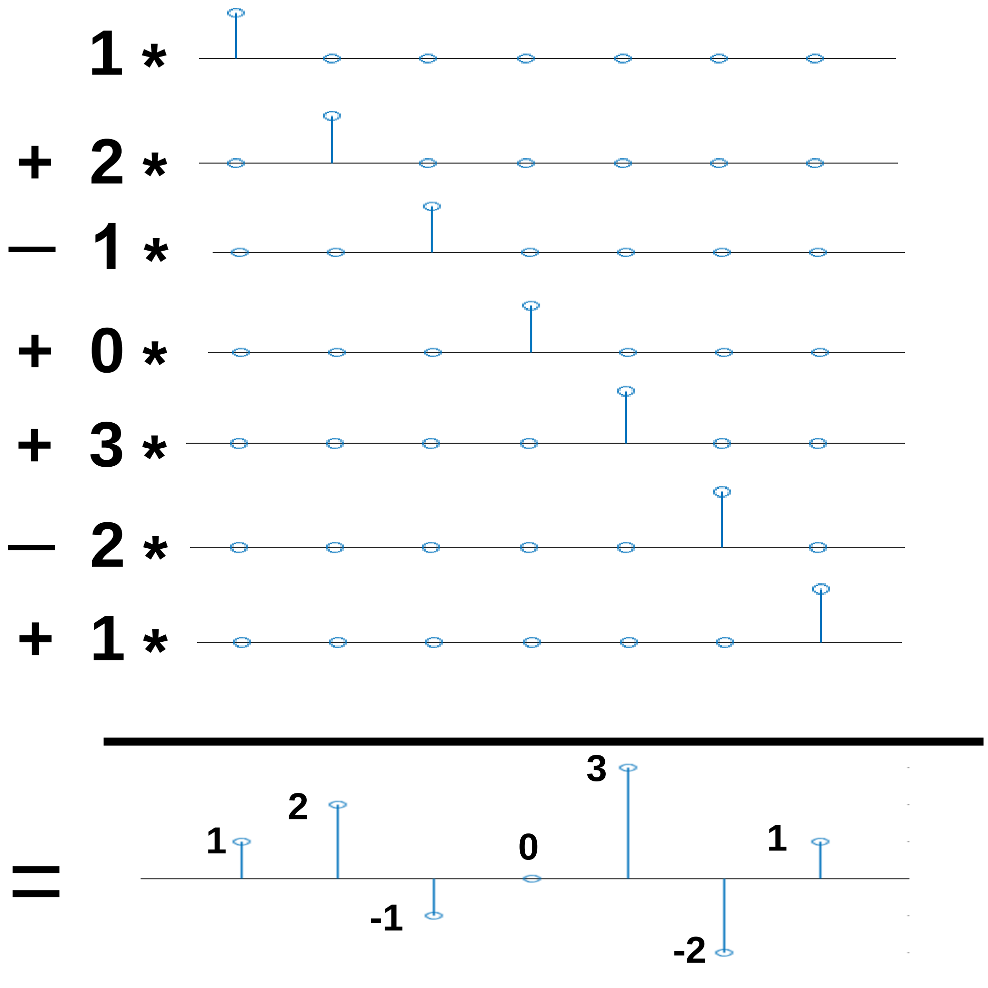
The SUI basis

(the *natural* basis)

Pick an arbitrary signal

… 1 2 -1 0 3 -2 1 …



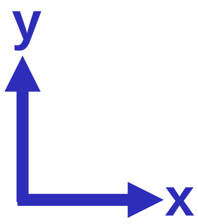


What is the natural basis for the analog signals?

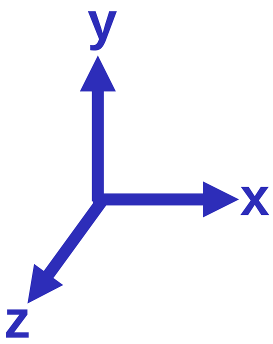
Dimension

The number of elements in the base
is called the *dimension*

2-space



3-space



The dimension of the vector space

- of all **digital** signals is denumerably infinite
- of all **analog** signals is nondenumerably infinite

Another basis

Vector fields can have more than one basis

For signals there is another important basis!

Let’s try to guess what it could be …

Fourier Demo

Fourier Series

In the demo we saw that many periodic analog signals can be written as the sum of Harmonically Related Sinusoids (HRSs)

If the period is T, the *frequency* is f = 1/T, the *angular frequency* is $\omega = 2\pi f = 2\pi / T$

$$s(t) = a_1 \sin(\omega t) + a_2 \sin(2\omega t) + a_3 \sin(3\omega t) + \ldots$$

But this can't be true for ***all*** periodic analog signals !

1. sum of sines is an odd function $s(-t) = -s(t)$
2. in particular, s(0) must equal 0

Similarly, it can't be true that ***all*** periodic analog signals obey

$$s(t) = b_0 + b_1 \cos(\omega t) + b_2 \cos(2\omega t) + b_3 \cos(3\omega t) + \ldots$$

Since this would give only even functions $s(-t) = s(t)$

We know that any (periodic) function can be written as the sum of an even (periodic) function and an odd (periodic) function

$$s(t) = e(t) + o(t) \quad \text{where } e(t) = (s(t) + s(-t))/2 \text{ and } o(t) = (s(t) - s(-t))/2$$

So Fourier claimed that all periodic analog signals can be written :

$$s(t) = a_1 \sin(\omega t) + a_2 \sin(2\omega t) + a_3 \sin(3\omega t) + \ldots + b_0 + b_1 \cos(\omega t) + b_2 \cos(2\omega t) + b_3 \cos(3\omega t) + \ldots$$

What does this say about the dimension of the subspace of periodic signals?

Fourier rejected

If Fourier is right, then-

the sinusoids are a basis for vector ***sub***space of periodic analog signals

Lagrange said that this can't be true –

not all periodic analog signals can be written as sums of sinusoids !

His reason –

the sum of continuous functions is continuous

the sum of smooth (continuous derivative) functions is smooth

His error –

the sum of a **finite number** of continuous functions is continuous

the sum of a **finite number** of smooth functions is smooth

Dirichlet came up with exact conditions for Fourier to be right :

- finite number of discontinuities in the period
- finite number of extrema in the period
- bounded
- absolutely integratable

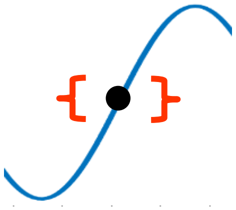
Why not polynomials?

The Taylor theorem tells us that functions (analog signals)
can be expanded in 1, t, t^2, t^3, …

So these are a basis of the space of analog signals
and there is an orthonormal version – the Legendre polynomials
and for discrete time the Szego polynomials

Why aren't these a useful basis for DSP ?

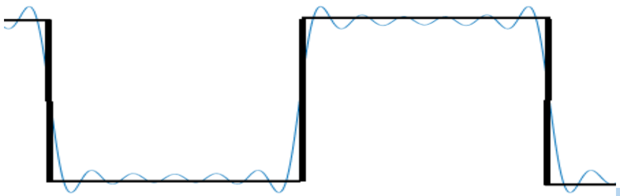
The Taylor expansion focuses on the area around a specific time t_0
The more basis functions we use
the larger the interval in which we know s(t)



In DSP we are interested in signals *at all times*
there is no special time t_0

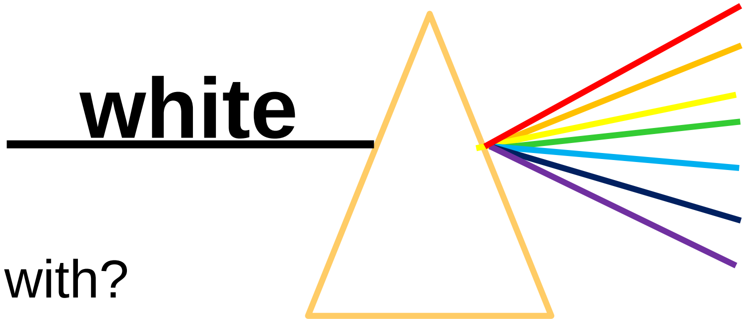
The Fourier expansion operates at all times simultaneously

The more basis functions we use
the better the approximation everywhere



Spectra

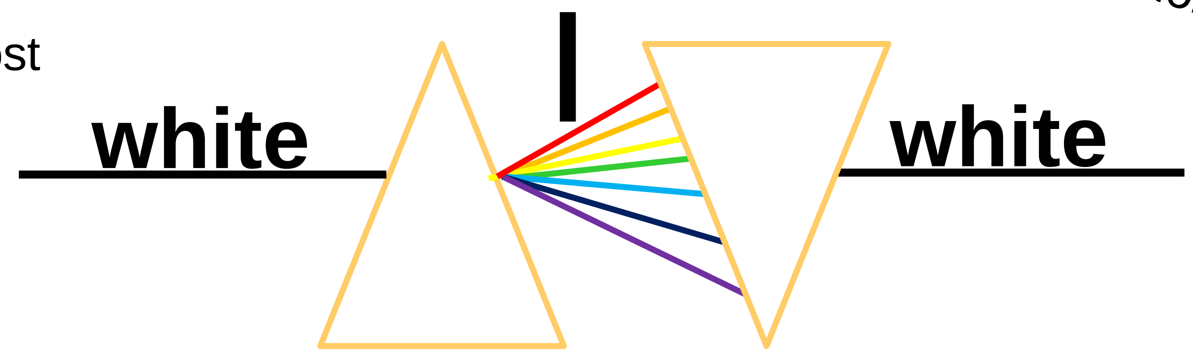
Newton used a prism and separates **white light** into **colors**



Does the prism paint the light
or were the colors there to begin with?

White light has all the colors!

spectrum = ghost



One can recreate the white light
but only if no color is blocked

Time and frequency domains

Vector spaces of signals have **two** important bases (SUIs and sinusoids)

And the *representations* (coefficients) of signals in these two bases give us two **domains**

Time domain (axis)

$s(t)$ s_n

Basis - **S**hifted **U**nit **I**mpulses

Frequency domain (axis)

$S(\omega)$ S_k

Basis - **sinusoids**

We use the same letter *capitalized* to stress that these are the same signal, just different representations

To go between the representations :

- analog signals - Fourier Transform FT/iFT
- digital signals - Discrete Fourier Transform DFT/iDFT

There is a *fast* algorithm for the DFT/iDFT called the FFT

Signals - recap

So, we now have to backtrack

The definitions of a signal as a function or sequence of time
are actually merely the representations of the signal
in the *time domain*

The signal also has a *frequency domain* representation

The signal is *more* than just its representation!

DSP is the art of working in both domains

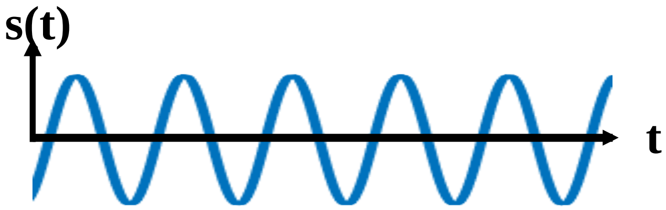
- some processing is easier in the time domain
- some processing is easier in the frequency domain

and we will have to go back and forth using the Fourier Transform

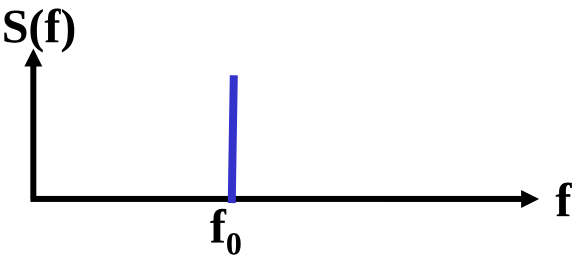
Simplest case – a sinusoid

We can now see the two representations using *regular* frequency

For an analog sinusoid $s(t) = A \sin(2\pi f_0 t)$

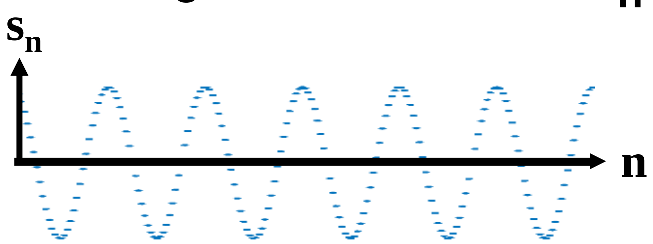


time domain

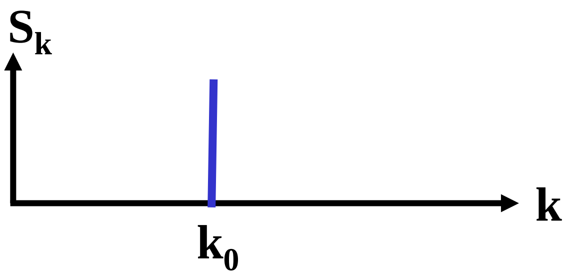


frequency domain

For a digital sinusoid $s_n = A \sin(2\pi k_0 n)$



time domain

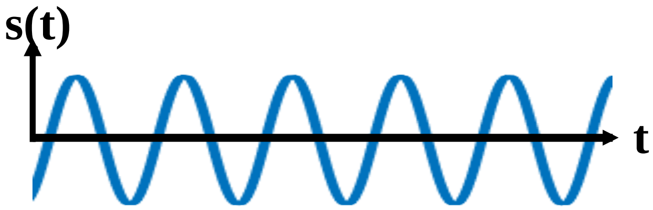


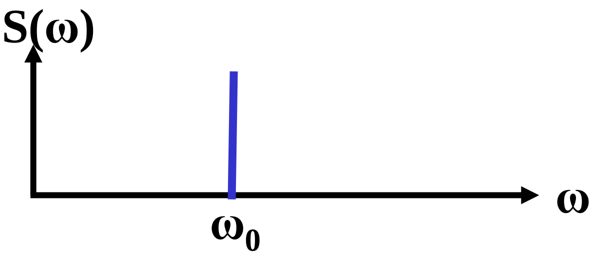
frequency domain

Simplest case – a sinusoid

We can now see the two representations using *angular* frequency

For an analog sinusoid $s(t) = A \sin(\omega_0 t)$

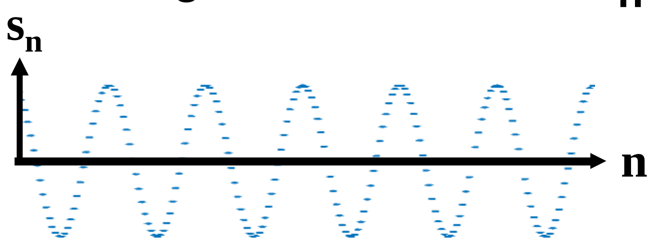


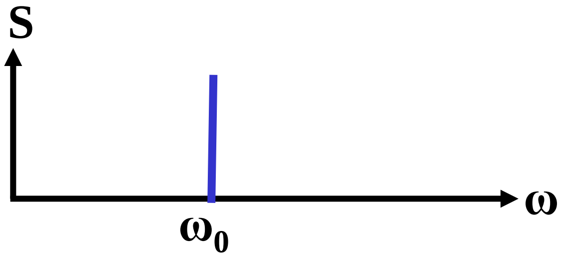


time domain

frequency domain

For a digital sinusoid $s_n = A \sin(\omega_0 n)$





time domain

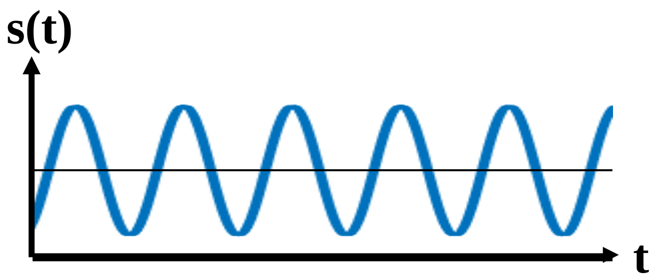
frequency domain

DC component

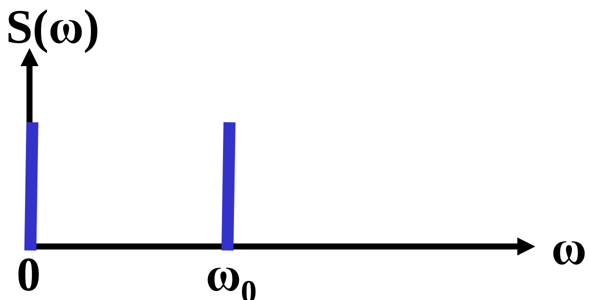
The DC component is the spectral component

at zero frequency   S(0) or S_0 (either regular or angular frequency)

$$s(t) = 1 + \sin(\omega_0 t)$$



time domain



frequency domain

So, we can give two interpretations to the DC component:

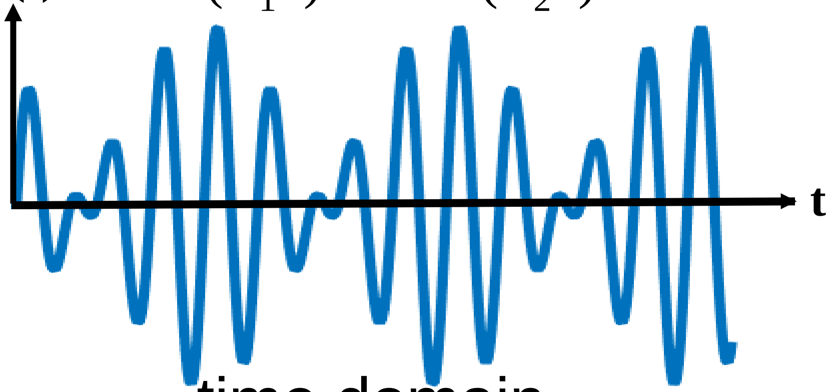
TIME DOMAIN INTERPRETATION : the average value of s(t) or s_n

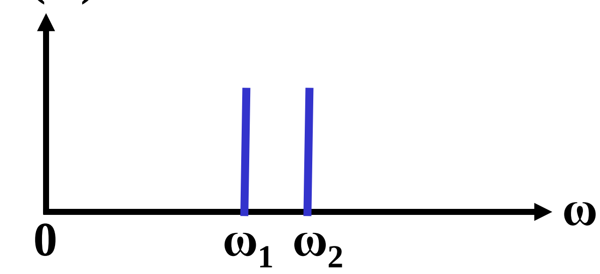
FREQUENCY DOMAIN INTERPRETATION : the value of S(0) or S_0

Two frequencies

What about the sum of two sinusoids (neither DC)?

$\mathbf{s(t)} = \sin(\omega_1 t) + \sin(\omega_2 t)$



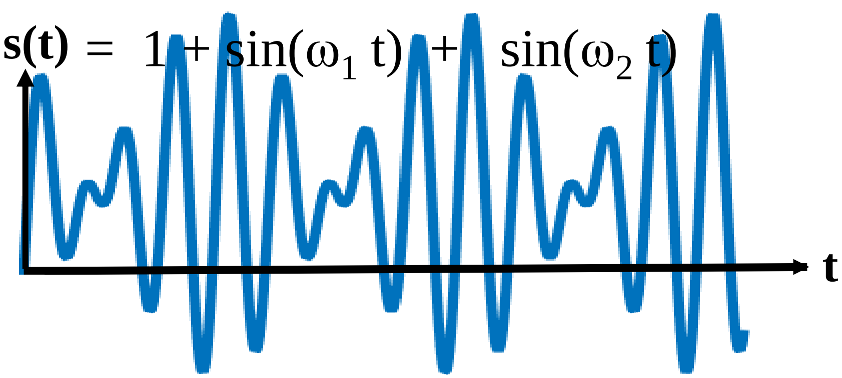


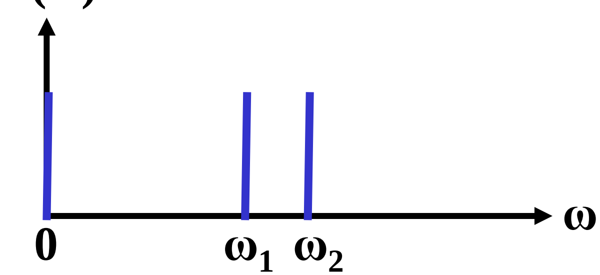
time domain

frequency domain

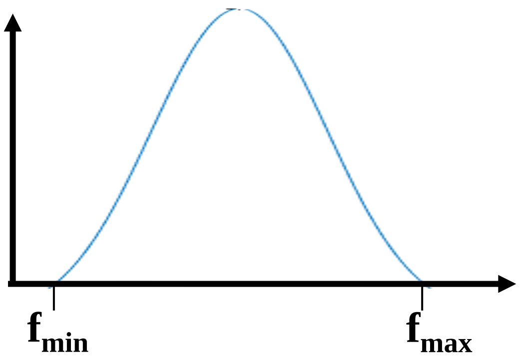
Two sinusoids *with* DC

$\mathbf{s(t)} = 1 + \sin(\omega_1 t) + \sin(\omega_2 t)$





Bandwidth



We can finally define *bandwidth*

Bandwidth is the *width* of the spectrum

$$BW = f_{max} - f_{min}$$

This is a *simplistic* definition

usually there will be some constant

depending on how we want to define the point when the spectrum is essentially zero

When the spectrum starts from DC

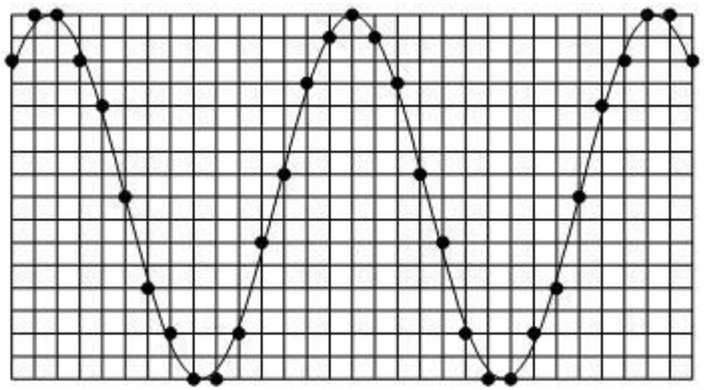
the BW is simply the highest frequency

Note that the bandwidth

of a single sinusoid (including DC) is zero!

Sampling again!

From an analog signal we can create a digital signal
by **SAMPLING**



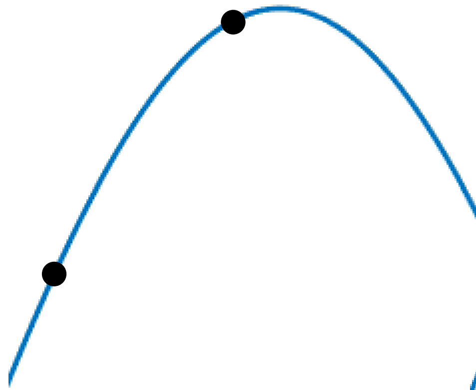
Under certain conditions
we can uniquely return to the analog signal !

Nyquist (Low pass) Sampling Theorem

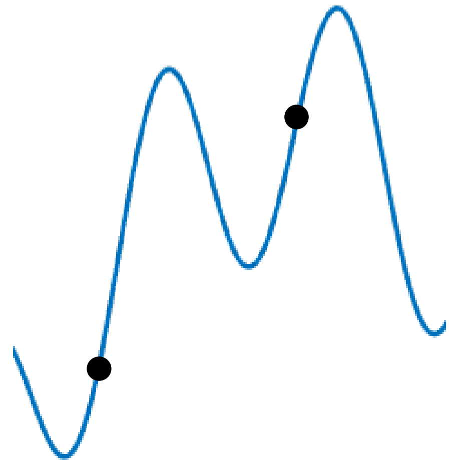
if the analog signal is BW limited and
has no frequencies in its spectrum above $F_{Nyquist}$
then sampling at above $2F_{Nyquist}$ causes no information loss

Does this make sense?

We understand how to interpolate if the analog signal looks like this :



But why can't the signal do this?



Because that requires frequencies above $F_{Nyquist}$!!

Does it really have to be > ?

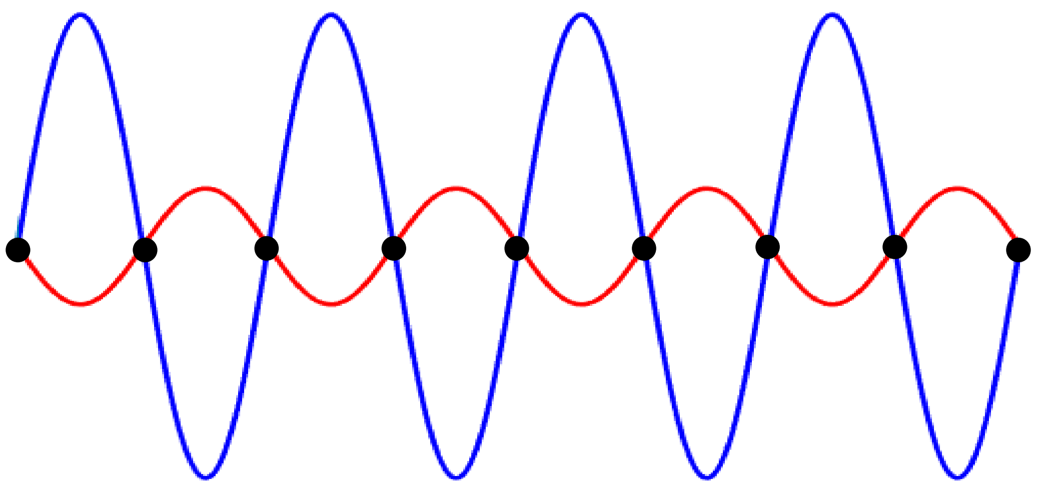
You might hear people casually say

that you need to sample at **twice** the highest frequency

WRONG!

You must sample at MORE than twice $F_{Nyquist}$!!

For example, here we sample at precisely twice $F_{nyquist}$



Aliasing

What happens if you sample at too low a frequency?

Wagon wheel demo helicopter demo

This is called aliasing!

The maximum allowed frequency is the **Nyquist** frequency

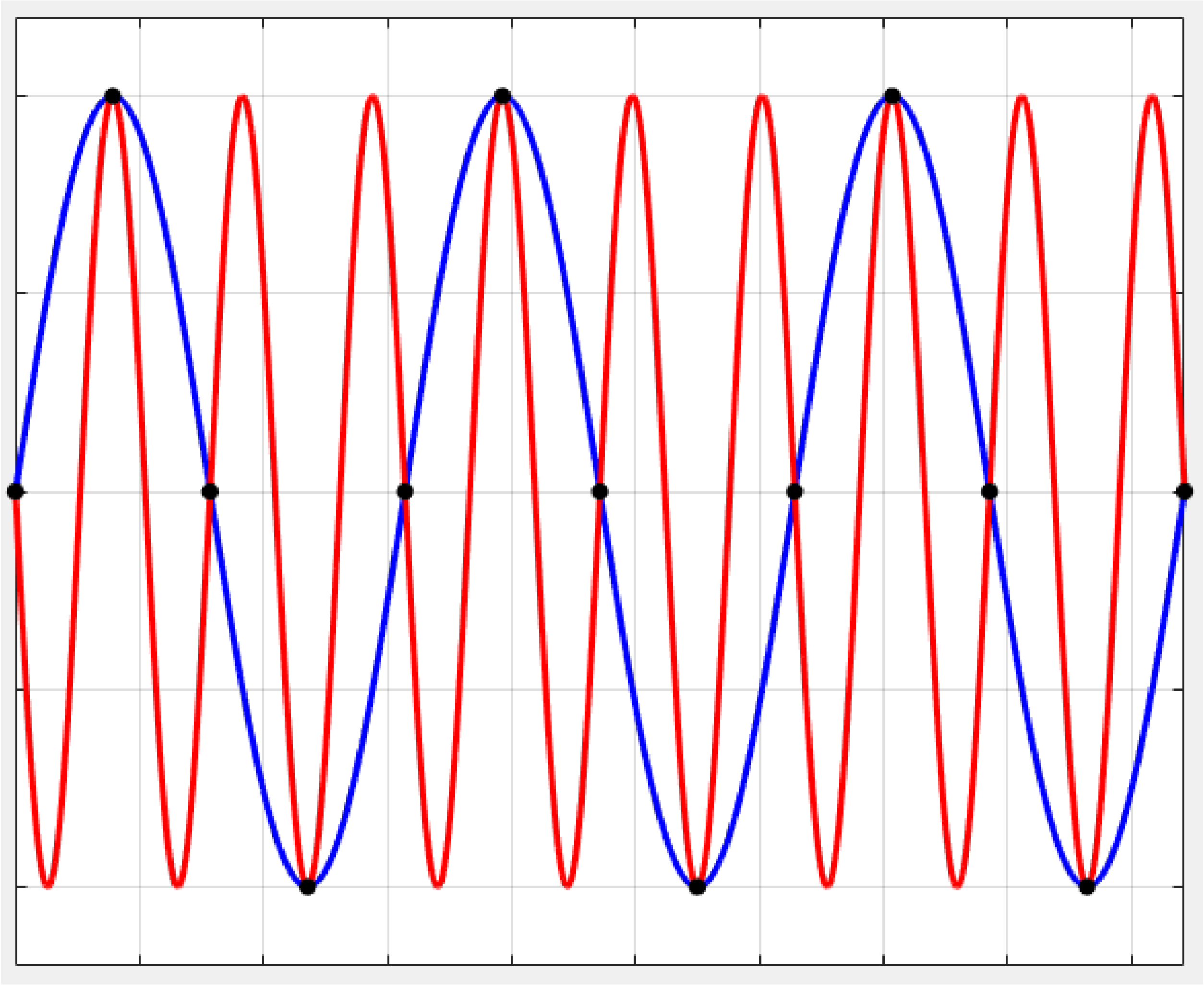
$$f_{\mathbf{Nyquist}} = f_{\mathbf{s}} / 2$$

When sampling we have to make sure
that there are not spectral components over $f_{\mathbf{Nyquist}}$

This is done using an **anti-aliasing filter**
that removes all energy in the spectrum over $f_{\mathbf{Nyquist}}$

(we'll learn about filters later …)

Aliasing in the time domain



What is the highest frequency?

In analog signal processing, frequency has no upper limit

But in DSP the Nyquist frequency is the highest one can go!

How much is it?

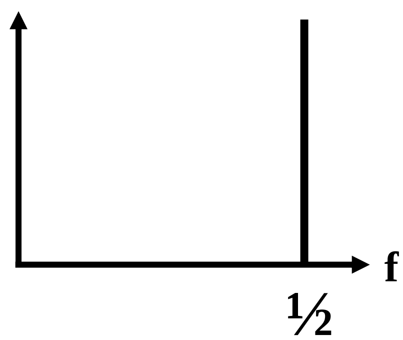
When sampling we define $n = t / t_s$ and thus $k = f / f_s$

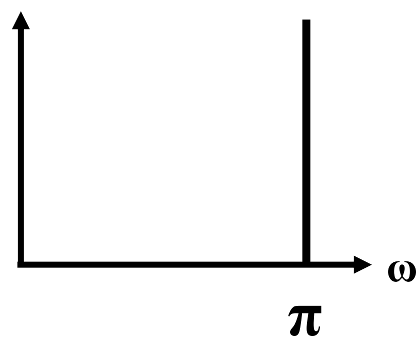
Thus, the highest k is $f_{max} = f_{Nyquist} / f_s = (½ f_s) / f_s = ½$

What about angular frequency?

The highest $\omega_{digital\text{-}max} = 2 \pi f_{max} = \pi$

All digital signals have finite bandwidth!

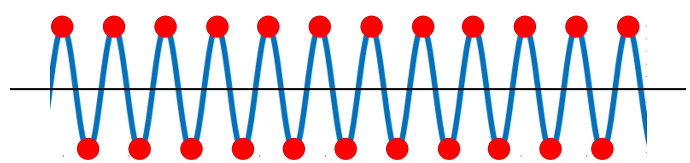




What is the Nyquist signal?

Sample a sine exactly twice per period

S = … +1 -1 +1 -1 …



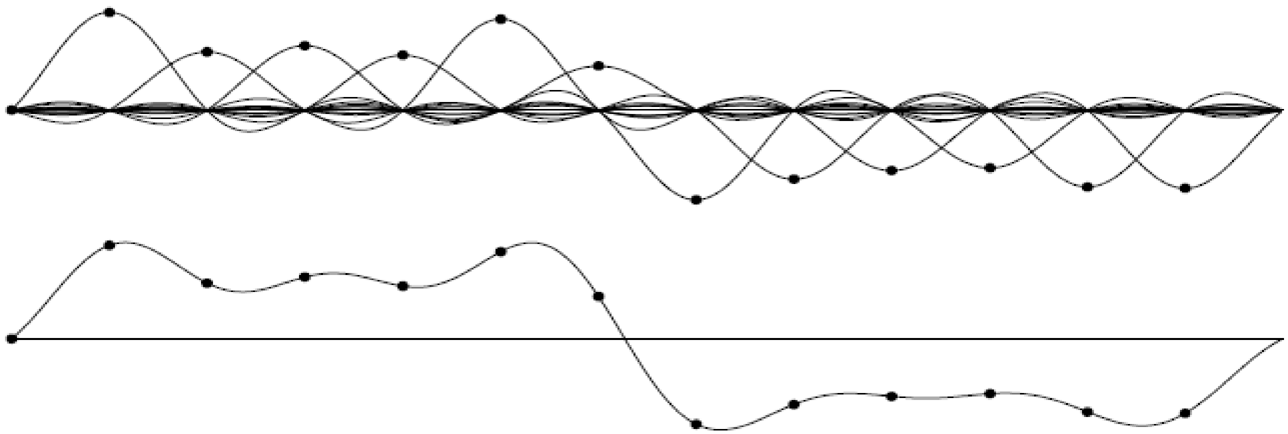
Reconstructing the signal

So, if we obey the sampling theorem and now have $s_\mathbf{n}$
how do we recover s(t) at some other time?

$$s(t) = \sum_{n=-\infty}^{\infty} s_n \, \mathrm{sinc}\Big(\pi f_s (t - n t_s)\Big)$$

frequency
$f_s/2 = f_{Nyquist}$
so, each sinc is zero at
neighboring sample points

In practice we only need a finite number of **sinc**s



Complex exponentials Negative frequencies

The Fourier series

$$s(t) = a_1 \sin(\omega t) + a_2 \sin(2\omega t) + a_3 \sin(3\omega t) + \ldots$$
$$+ b_0 + b_1 \cos(\omega t) + b_2 \cos(2\omega t) + b_3 \cos(3\omega t) + \ldots$$

has a basis consisting of 2 *different* kinds of signal $\sin(k\omega t)$ and $\cos(k\omega t)$

Can we find a series with a single type of basis ?

$$s(t) = c_0 + c_1 \sin(\omega t + \Phi_1) + c_2 \sin(2\omega t + \Phi_2) + \ldots$$

where $c_k = \sqrt{(a_k^2 + b_k^2)}$ and $\Phi_k = \arctan_4 (b_k / a_k)$

works, but isn't a basis – it contains nondenumerable number of signals!

Substituting $\cos(k\omega t) = \frac{1}{2} (e^{ik\omega t} + e^{-ik\omega t})$ and $\sin(k\omega t) = \frac{1}{2\mathbf{i}}(e^{ik\omega t} - e^{-ik\omega t})$

we find the Fourier Series in terms of complex exponentials

$$s(t) = \sum_{k=-\infty}^{\infty} S_k \, e^{ik\omega t}$$

For example

$$a_1 \sin(\omega t) + b_1 \cos(\omega t) = \frac{a_1}{2\mathbf{i}}(e^{i\omega t} - e^{-i\omega t}) + \frac{b_1}{2} (e^{i\omega t} + e^{-i\omega t}) = \underbrace{(\frac{a_1}{2\mathbf{i}} + \frac{b_1}{2})}_{S_{+1}} e^{i\omega t} + \underbrace{(-\frac{a_1}{2\mathbf{i}} + \frac{b_1}{2})}_{S_{-1}} e^{-i\omega t}$$

Handling the problems

This is aesthetically pleasing, but raises two problems:

1. the basis functions $e^{ik\omega t}$ are complex
   and thus not *signals* at all !

2. the frequencies $k\omega$ can be negative
   what does -2 cycles per second mean ?

Using complex exponentials and negative frequencies
so simplifies the mathematics
that we will do anything to allow it

demo

1. We are understand that the true signals are real
   – they are simply **Re**$(e^{ik\omega t})$
   At the end of the calculations we look at the real part

2. The negative frequencies are the same as the positive ones
   only the phase is different (see demo)

FS, FT, DFT – the long journey

Fourier **S**eries (periodic analog signal → $\aleph_0$ coefficients)

$$\mathbf{s(t)} = \sum_{k=-\infty}^{\infty} S_k \, e^{-i\,k\omega\,t}$$

Fourier **T**ransform (general analog signal → spectrum)

$$\mathbf{S(\omega)} = \int_{-\infty}^{\infty} s(t)\, e^{-i\omega t}\, dt$$

$$\mathbf{s(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\, e^{i\omega t}\, d\omega$$

Discrete **F**ourier **T**ransform (finite digital signal → digital spectrum)

$$\mathbf{S_k} = \sum_{n=0}^{N-1} W_N^{nk}\, s_n$$

$$\mathbf{s_n} = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-nk}\, S_k$$

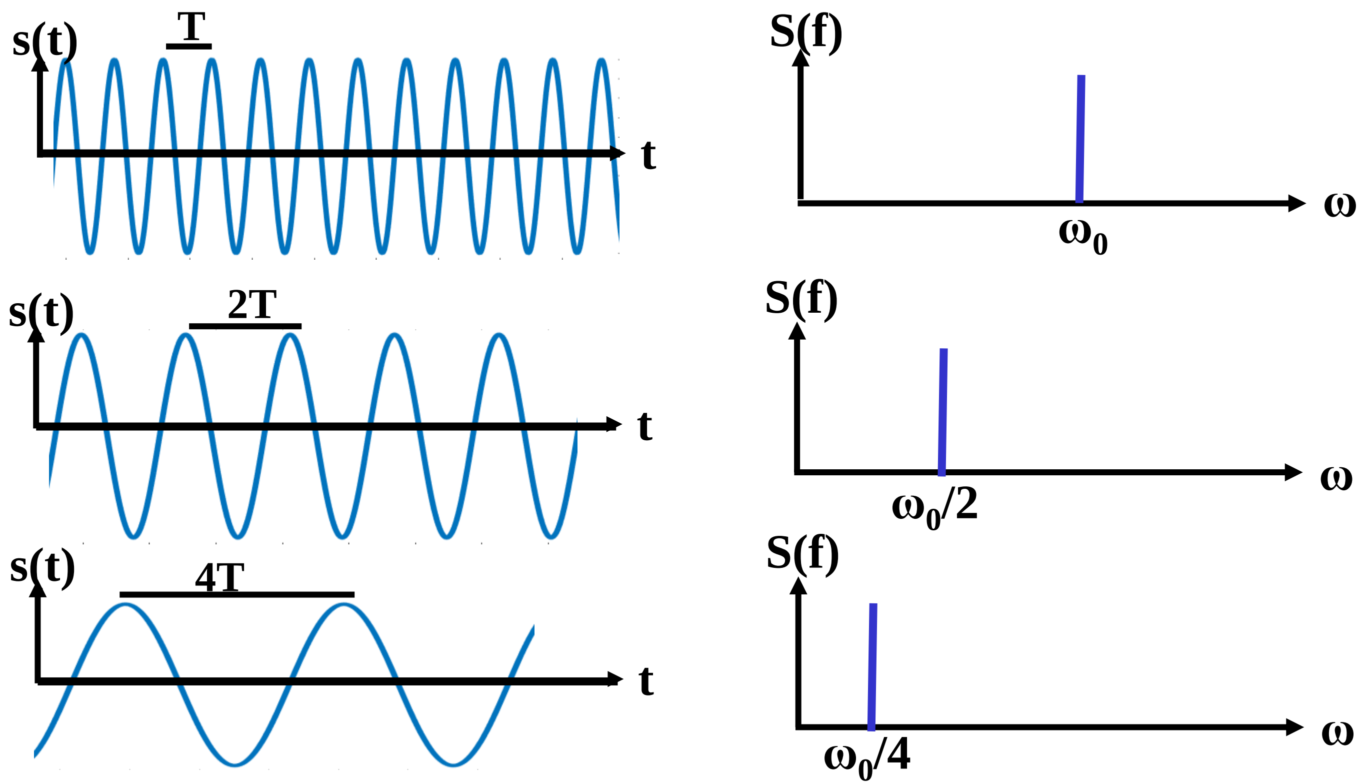
Actually, this is the **Short Time** DFT

From FS to FT

The Fourier series transforms a periodic analog signal
into a denumerable set of coefficients

What about non-periodic analog signals?

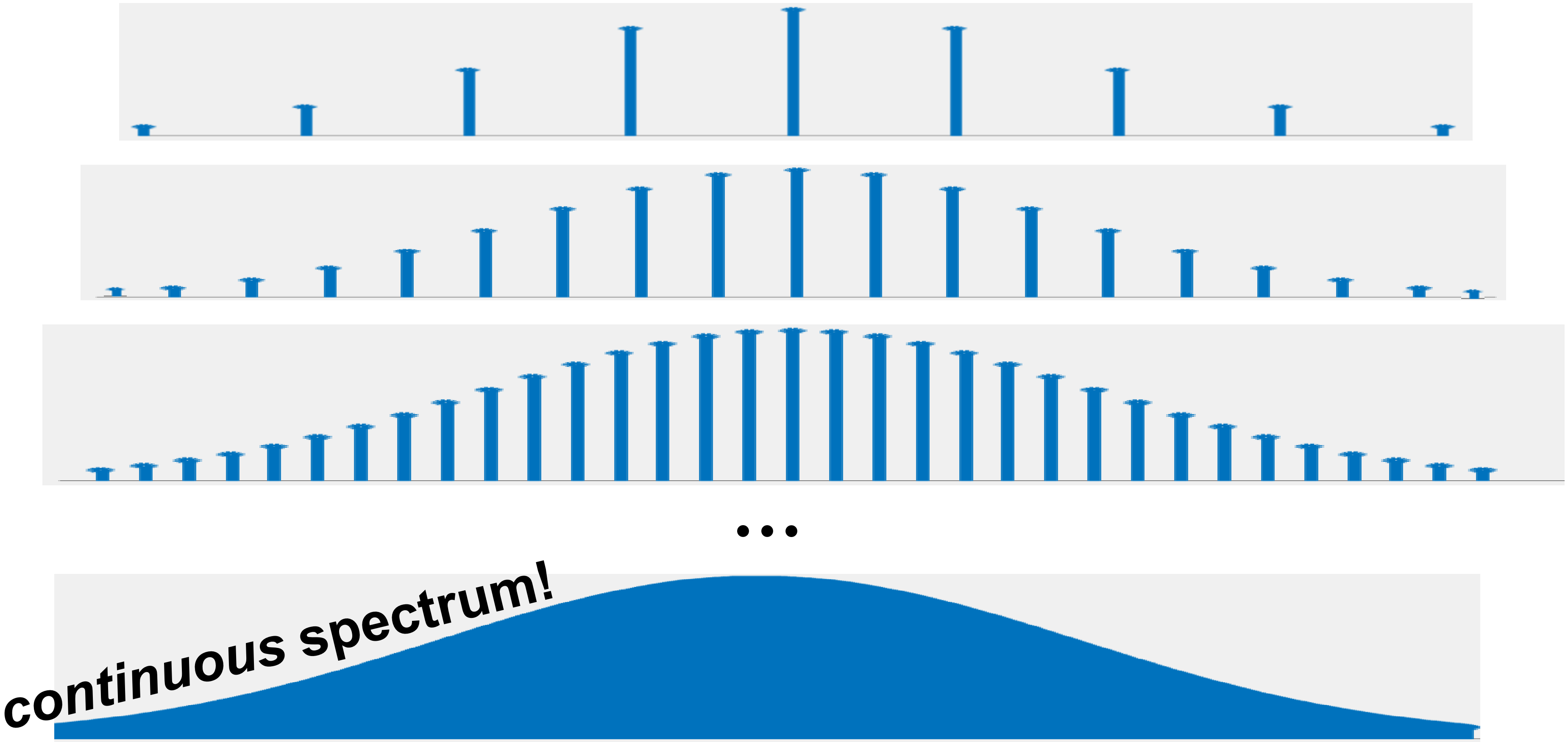
A non-periodic signal is the limit of a periodic one with period $T \to \infty$
so the base frequency $f \to 0$!



The spectrum in the limit

What about a periodic signal that is not a sinusoid?

i.e., has lots of frequencies in its spectrum ? (all multiples of ω_0)



How can you recognize a periodic signal from its spectrum?

The Fourier Transform

So, the *sum* becomes an *integral*

and nonperiodic analog signals have Fourier *Transforms*

which are functions s(t) → S(**ω**)

S(**ω**) = FT(s(t)) s(t) = iFT(S(**ω**)) (iFT = FT^{-1})

The precise definitions are the integrals

$$S(\omega) = \mathrm{FT}\Big(s(t)\Big) = \int_{t=-\infty}^{\infty} s(t)\, e^{-\mathrm{i}\omega t}\, dt$$

$$s(t) = \mathrm{FT}^{-1}\Big(S(\omega)\Big) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} S(\omega)\, e^{\mathrm{i}\omega t}\, d\omega$$

Of course FT FT^{-1} = FT^{-1} FT = 1

which means that the product of the constants must be 1 / 2π

but there are various conventions as to how to do this

But we're interested in DSP!

Sampling the analog signal in the time domain $n = t / t_s$

the integral becomes a sum

i.e., sampling in the frequency domain $k = f / f_s = \omega / \omega_s$

Why don't we need 2 ks, one for regular and one for angular frequency?

$$s(t) \rightarrow s_n \quad \leftrightarrow \quad S(\omega) \rightarrow S_k$$

We won't prove this here – see the textbook!

but note that $t_s = 1/f_s$ so $f\, t = f/fs * t/ts = k\, n$

So the integral over $s(t)\, e^{i\, 2\, \pi\, f\, t}$

will turn into a sum over $s_n\, e^{i\, 2\, \pi\, n\, k}$

$$S_k = \sum_{n=0}^{N-1} e^{-i \frac{2 \pi n k}{N}} s_n$$

The product of coefficients has to be 1/N

and we are following a (bad) convention

$$s_n = \frac{1}{N} \sum_{k=0}^{N-1} e^{i \frac{2 \pi n k}{N}} S_k$$

The DFT

So, the transformation now is the **D**iscrete **F**ourier **T**ransform

$S_k = \text{DFT}(\, s_n \,)$ $\quad\quad$ $s_n = \text{iDFT}(\, S_k \,)$ $\quad$ ($\text{iDFT} = \text{DFT}^{-1}$)

We will deal with digital signals with N times

Yes, signals are defined for all times

but we don't care about all other values

you can think that all other values are zero

or that the signal repeats over and over again

The precise form of the DFT and iDFT for finite N is:

$$\mathbf{S_k} = \sum_{n=0}^{N-1} e^{-i\frac{2\pi n k}{N}} s_n \qquad \mathbf{k = 0 \ldots N\text{-}1}$$

$$\mathbf{s_n} = \frac{1}{N}\sum_{k=0}^{N-1} e^{i\frac{2\pi n k}{N}} S_k \qquad \mathbf{n = 0 \ldots N\text{-}1}$$

Why must there be the same number of **k**s as **n**s ?

DFT is a transformation

If we think of the digital signals as vectors

then the DFT and iDFT are (unitary transformation) matrices

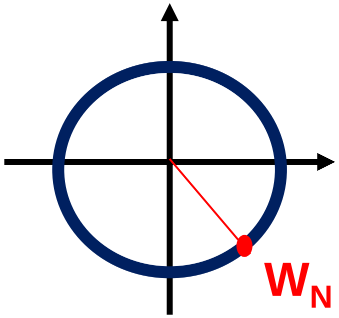
They change the representation from the SUI basis to the HRS one

We call the matrix **W**

So, we can write the DFT like this:

$$\begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N-1} \end{pmatrix} = \begin{pmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,N-1} \\ W_{1,0} & W_{1,1} & \cdots & W_{1,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ W_{N-1,0} & W_{N-1,1} & \cdots & W_{N-1,N-1} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{pmatrix}$$

The Nth root of unity

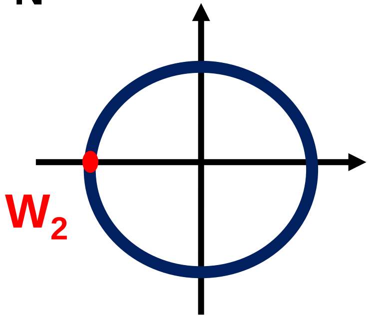


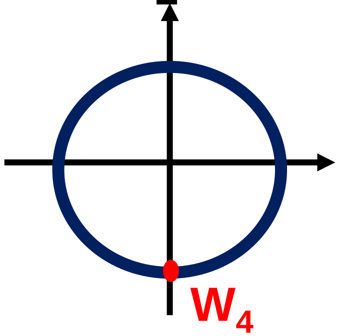
There's a better way of writing this

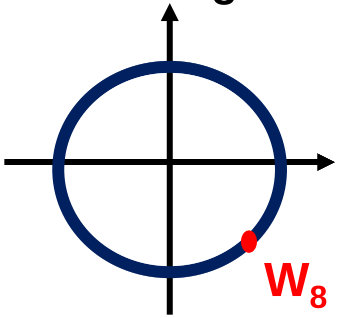
We define the N^{th} root of unity W_N, i.e., a number such that W_N^N **= 1**

It is also called the twiddle factor (we'll see why later!)

$W_N = e^{-i\frac{2\pi}{N}}$ for example, $W_2 = -1$ $W_4 = -i$ $W_8 = -\frac{\sqrt{2}}{2}(1+i)$







Now

$$S_k = \sum_{n=0}^{N-1} W_N^{nk} s_n \qquad k = 0 \ldots N-1$$

$$s_n = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-nk} S_k \qquad n = 0 \ldots N-1$$

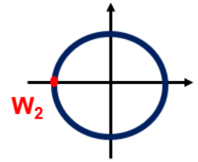
Example : N=2

Let's try one

$$\begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N-1} \end{pmatrix} = \begin{pmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,N-1} \\ W_{1,0} & W_{1,1} & \cdots & W_{1,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ W_{N-1,0} & W_{N-1,1} & \cdots & W_{N-1,N-1} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{pmatrix}$$

For N=2 we have 2 signal values in the time domain s_0 s_1

$$W_2 = e^{-i\frac{2\pi}{2}} = e^{-i\pi} = -1$$



and in the frequency domain $S_0 = s_0 + s_1$ $S_1 = s_0 - s_1$

$$\begin{pmatrix} S_0 \\ S_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} \qquad \begin{pmatrix} s_0 \\ s_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} S_0 \\ S_1 \end{pmatrix}$$

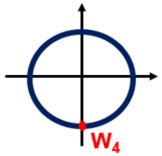
Example : N=4

Let's try another one

$$\begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N-1} \end{pmatrix} = \begin{pmatrix} W_{0,0} & W_{0,1} & \cdots & W_{0,N-1} \\ W_{1,0} & W_{1,1} & \cdots & W_{1,N-1} \\ \cdots & \cdots & \cdots & \cdots \\ W_{N-1,0} & W_{N-1,1} & \cdots & W_{N-1,N-1} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N-1} \end{pmatrix}$$

For N=4 we have 4 signal values in the time domain s_0 s_1 s_2 s_3

$$W_4 = e^{i\frac{2\pi}{4}} = e^{i\pi/4} = -i$$



$$\underline{\underline{W_4}} = \begin{pmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

The W matrix in general

$$
\begin{array}{c} \\ n=0 \\ 1 \\ 2 \\ 3 \\ \\ \\ N-1 \end{array}
\begin{array}{c}
\begin{array}{cccccc} k=0 & 1 & 2 & 3 & & n-1 \end{array} \\
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & W_N & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\
1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\
1 & W_N^3 & W_N^6 & W_N^9 & \cdots & W_N^{3(N-1)} \\
1 & -1 & 1 & -1 & \cdots & \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & W_N^{-1} & W_N^{-2} & W_N^{-3} & \cdots & W_N^{1-N}
\end{pmatrix}
\end{array}
$$

The first row is all 1s

so S_0 (the DC component) is the sum of all values

it would be the average if we used a different convention

The second row is powers of W_N

The third row is powers of W_N^2

Row N/2 + 1 is +1 -1 +1 -1

The first column is all 1s

This matrix is not orthogonal but unitary – why?

Another meaning for frequency

Frequency is well defined for sinusoids

$s(t) = A \sin(\omega t + \phi)$ $s_n = A \sin(\omega n + \phi)$

A amplitude
ω (angular) frequency
Φ phase

but all other signals have many - an entire *spectrum* of frequencies

There is an intuitive feeling
that at every time a signal
can have a different *instantaneous* amplitude and frequency
Can we find $s(t) = A(t) \sin(\omega(t)\, t)$ for all signals?

Does that make sense? It can't be unique!

Given any $s(t)$ pick an arbitrary $\omega(t)$ and divide $A(t) = s(t) / \sin(\omega(t)\, t)$

A consistent meaning of instantaneous amplitude and frequency
can be obtained by a different transform (not Fourier!)

This new transform only works under 2 conditions

- the signal has finite bandwidth (which real signals should)
- the signal has no DC component – i.e., its time average is zero

Hilbert transform

The instantaneous (analytical) representation

- $x(t) = A(t) \cos (\Phi(t)) = A(t) \cos (\omega_c t + \phi(t))$
- $A(t)$ is the *instantaneous amplitude*
- $\phi(t)$ is the *instantaneous phase*

$\boldsymbol{\omega_c}$ center frequency
carrier frequency

This is used in information transmission

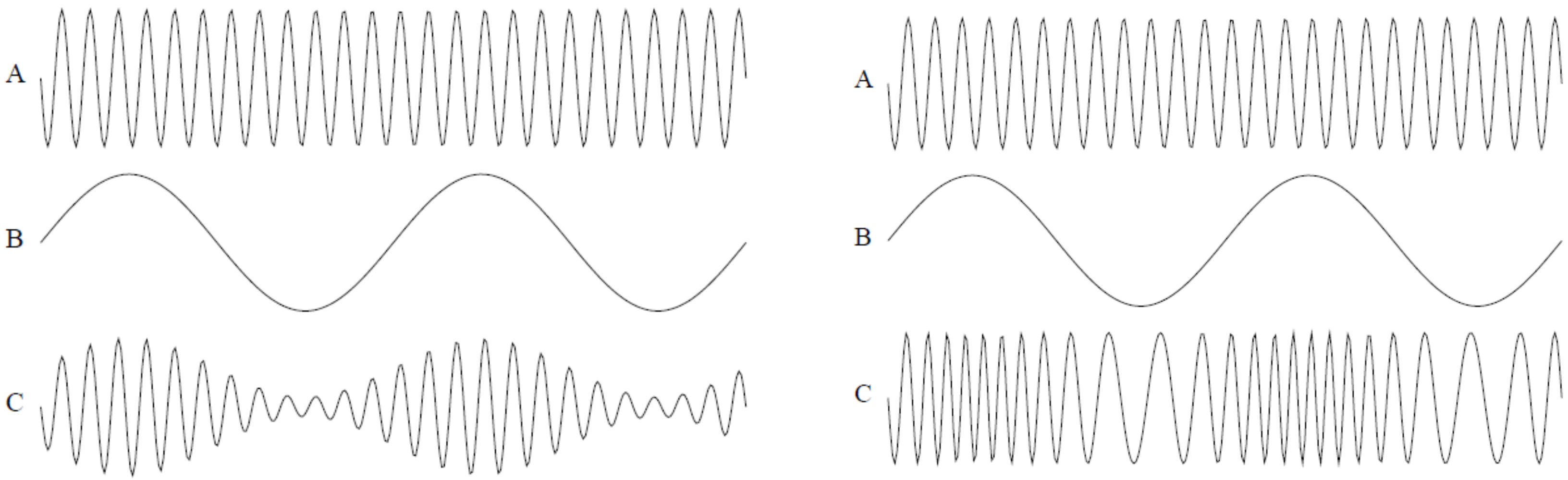
- **A**mplitude **M**odulation $x(t) = A(t) \cos (\omega_c t)$
- **P**hase **M**odulation $x(t) = A \cos (\omega_c t + \phi(t))$
- **F**requency **M**odulation $x(t) \neq A \cos (\omega(t) t)$

Why not?

Frequency is the derivative of phase

- $\Phi(t) = \omega_0 t$ then $\omega(t) = \omega_0$
- $\Phi(t) = \omega_c t + \phi(t)$ then $\omega(t) = \omega_c + \frac{d}{dt}\phi(t)$

AM and FM



Modulation means changing some parameter of a signal
so that it carries *information*

Here we change the sinusoid's *amplitude* or *frequency (phase)*

Hilbert transform

The Hilbert transform is a 90 degree *phase shifter*

$$\hat{H} \cos(\Phi(t)) = \sin(\Phi(t))$$

Hence

- $x(t) = A(t) \cos(\Phi(t))$
- $y(t) = \hat{\mathbf{H}}\, x(t) = A(t) \sin(\Phi(t))$
- $A(t) = \sqrt{x^2(t) + y^2(t)}$
- $\Phi(t) = \arctan_4\left({}^{y(t)}/_{x(t)} \right)$

this is equivalent to shifting every spectral component separately by 90^o

The instantaneous frequency is the derivative of the instantaneous frequency

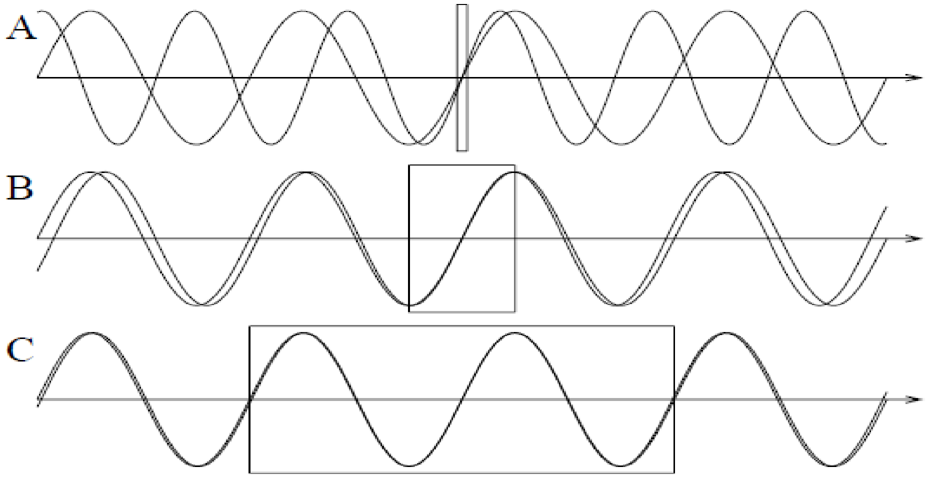
Uncertainty Theorem

When you see a sinusoid for a long time Δt

it is easy to measure its frequency : $N_{cycles} / \Delta t$

But when you only see it for a short time

the uncertainty in frequency $\Delta\omega$ is large



The uncertainty theorem (well-known in quantum mechanics) says

$$\Delta\omega \, \Delta t \; > \; \text{constant}$$

One more transform

What is a transform in mathematics?

An operation that transforms some object into a similar object while not losing information

For example

- the FT transforms a function into a function
- The DFT transforms a sequence into a sequence

The Fourier series is not a transform

since it converts a function into a sequence

In DSP we frequently use the z transform, although

- it is not a transform (it converts a sequence into a function)
- the name z is arbitrary

The zT is only defined for **digital** signals   why?

Let's start with something you already know - generating functions

Generating functions

In math we have the strong tools of *analysis (calculus)*
to study functions

For example
we can learn about a function by looking at its derivatives

But we have very few general mechanisms to handle sequences
(which is not good for DSP!)

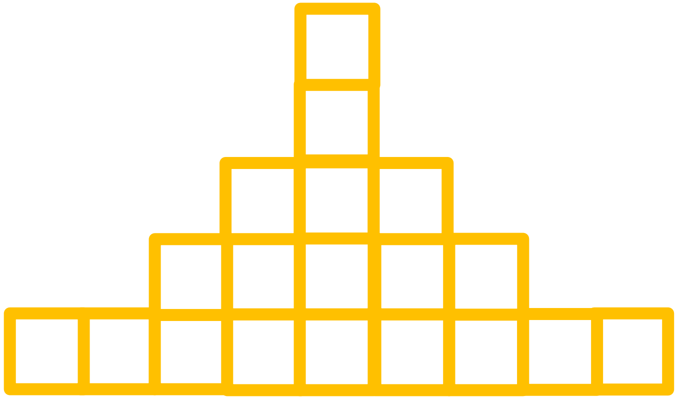
So, mathematicians came up with a way
to convert a sequence into a function (the generating function)
and by studying the function we learn a lot about the sequence!

Given a sequence s_n where $n = 0 \ldots \infty$
we define the generating function to be the power series

$$s(x) \equiv \sum_{n=0}^{\infty} s_n x^n$$

Riddle?

Why is this sculpture in Jerusalem called the golden sculpture?



Example – Fibonacci (1)

We all know the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, …
which is recursively defined by

$$f_0 = 1 \qquad f_1 = 1 \qquad f_n = f_{n-1} + f_{n-2}$$

It is well known that the ratio of two consecutive terms
tends to the golden ratio $\gamma \equiv \frac{1+\sqrt{5}}{2} = \cos^{-1}\left(\frac{\pi}{5}\right)$ ≈ 1.618

(note that γ-1 = 1/ γ !) $\gamma' = \frac{1-\sqrt{5}}{2}$ ≈ -0.618

There is a nonrecursive formula for the n^th term!

The generating function is

$$f(x) = \sum_{n=0}^{\infty} f_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5 + \ldots$$

We want to study this function

BTW, there is a nonrecursive formula for the N^th hexadecimal digit of π!

Example – Fibonacci (2)

We'll use here for the first time some tricks
that we'll be using again and again, so let's do it slowly!

sum from n=2 since the left term has n-2

$$\sum_{n=2}^{\infty} f_n x^n = \sum_{n=2}^{\infty} f_{n-1} x^n + \sum_{n=2}^{\infty} f_{n-2} x^n$$

$$= x \sum_{n=2}^{\infty} f_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} f_{n-2} x^{n-2}$$

substitute m = n-1 substitute m = n-2

$$= x \sum_{n=1}^{\infty} f_n x^n + x^2 \sum_{n=0}^{\infty} f_n x^n$$

rename m back to n

$$f(x) - f_0 x^0 - f_1 x^1 = x\left(f(x) - f_0 x^0\right) + x^2 f(x)$$

$$f(x) - 1 - x = f(x)x - x + f(x)x^2$$

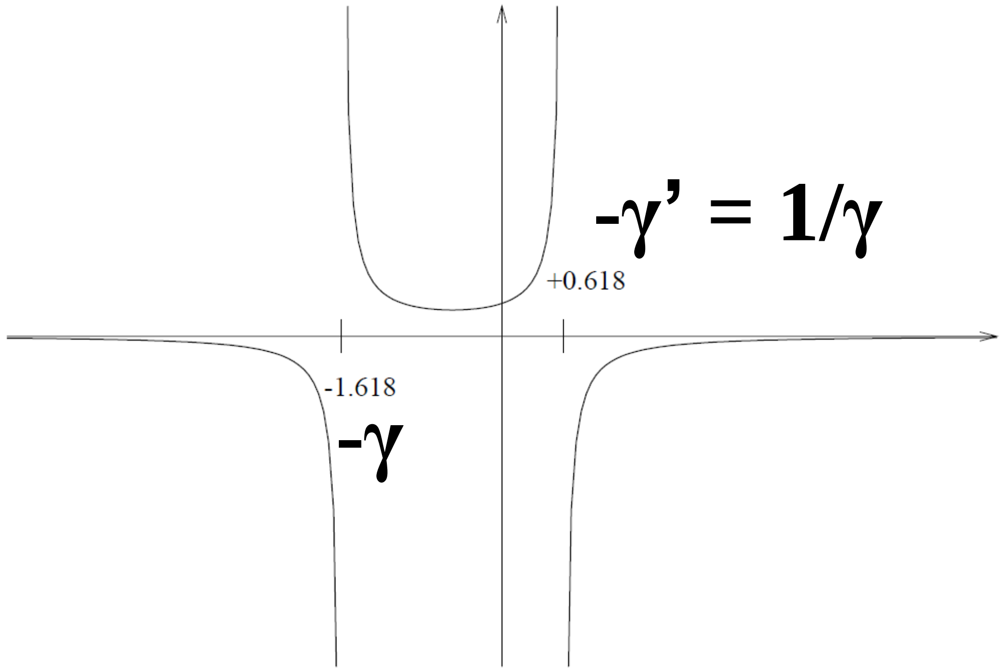
Example – Fibonacci (2)

So

$$f(x) = \frac{1}{1 - x - x^2}$$

Using the techniques learned in elementary calculus we can sketch it

note: $x^2 + x - 1 = 0$ has roots $-\gamma' = 1/\gamma$ and $-\gamma$



Example – Fibonacci (2)

Using a *partial fraction expansion*

$$f(x) = \frac{1}{1 - x - x^2} = \frac{A}{x + \gamma'} + \frac{B}{x + \gamma}$$

so

$$1 = A(x + \gamma) + B(x + \gamma')$$

substituting the roots we find

$$A = -B = \frac{1}{\sqrt{5}}$$

and so

$$f(x) = \frac{1}{\sqrt{5}} \left(\frac{1}{x + \gamma'} - \frac{1}{x + \gamma} \right) = \frac{1}{\sqrt{5}} \left(\frac{\frac{1}{\gamma'}}{1 + \frac{x}{\gamma'}} - \frac{\frac{1}{\gamma}}{1 + \frac{x}{\gamma}} \right)$$

and comparing to the sum of a geometric series we can find :

$$f_n = \frac{1}{\sqrt{5}} \left(\gamma^{n+1} - (\gamma')^{n+1} \right)$$

How can this be an integer?

Why does the ratio f_{n+1} / f_n approach γ ?

From generating function to z transform

To use this technique for digital signals we define the z Transform

$$S(z) = \mathrm{zT}(s_n) = \sum_{n=-\infty}^{\infty} s_n z^{-n}$$

There are 3 changes as compared to the generating function

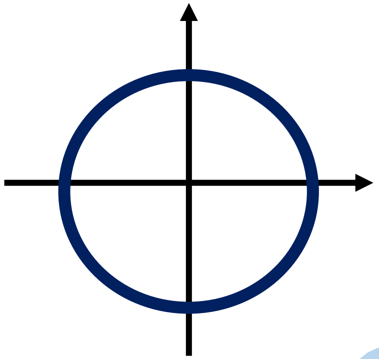
1. sum over n=-∞ … +∞ (sum – not integral! only for digital signals)
2. minus in exponent (convention)
3. z instead of x - since z is a **complex** variable

The zT allows us to use the even more powerful techniques of *complex analysis*

The zT is defined over the complex plane

Each point in the plane represents a signal

$$S_n = z^n$$

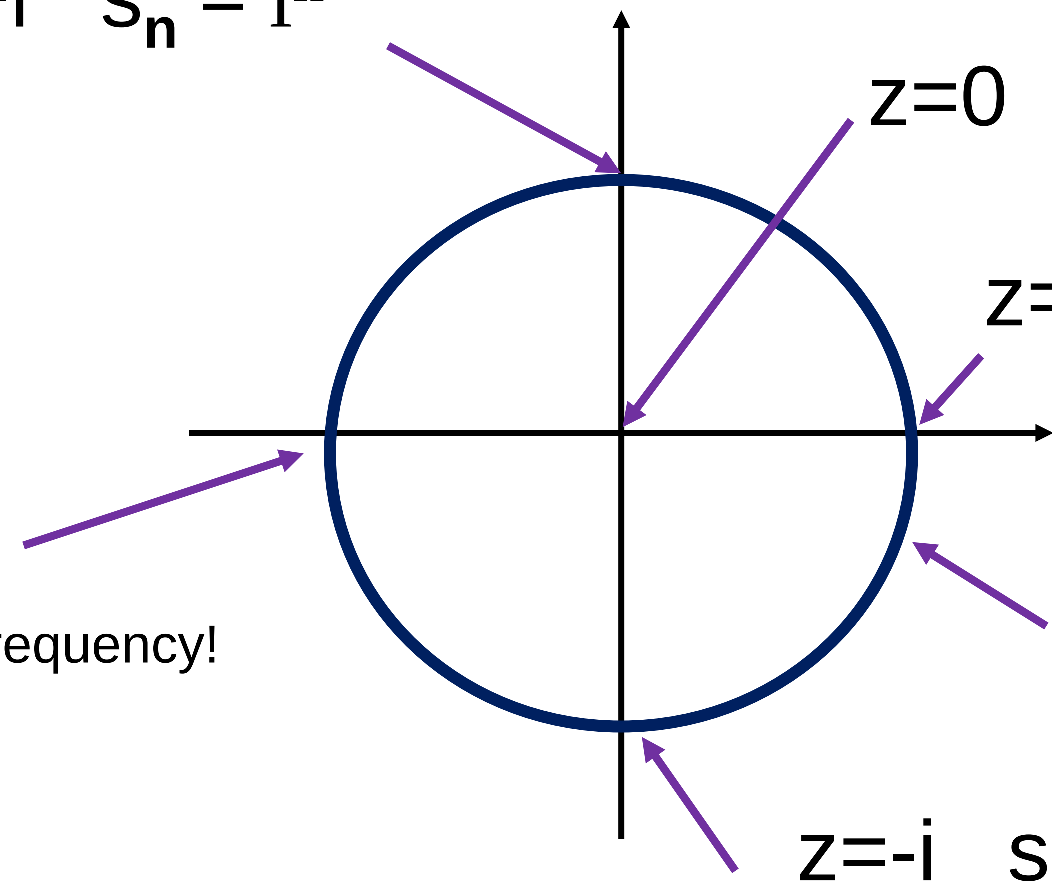


The z plane

$z=+i \quad s_n = i^n$

$z=0 \quad s_n=0$

$z=1 \quad s_n=1$ (DC)



$z = -1$

Nyquist frequency!

unit circle

(all z with $|z|=1$)

$z=-i \quad s_n = (-i)^n$

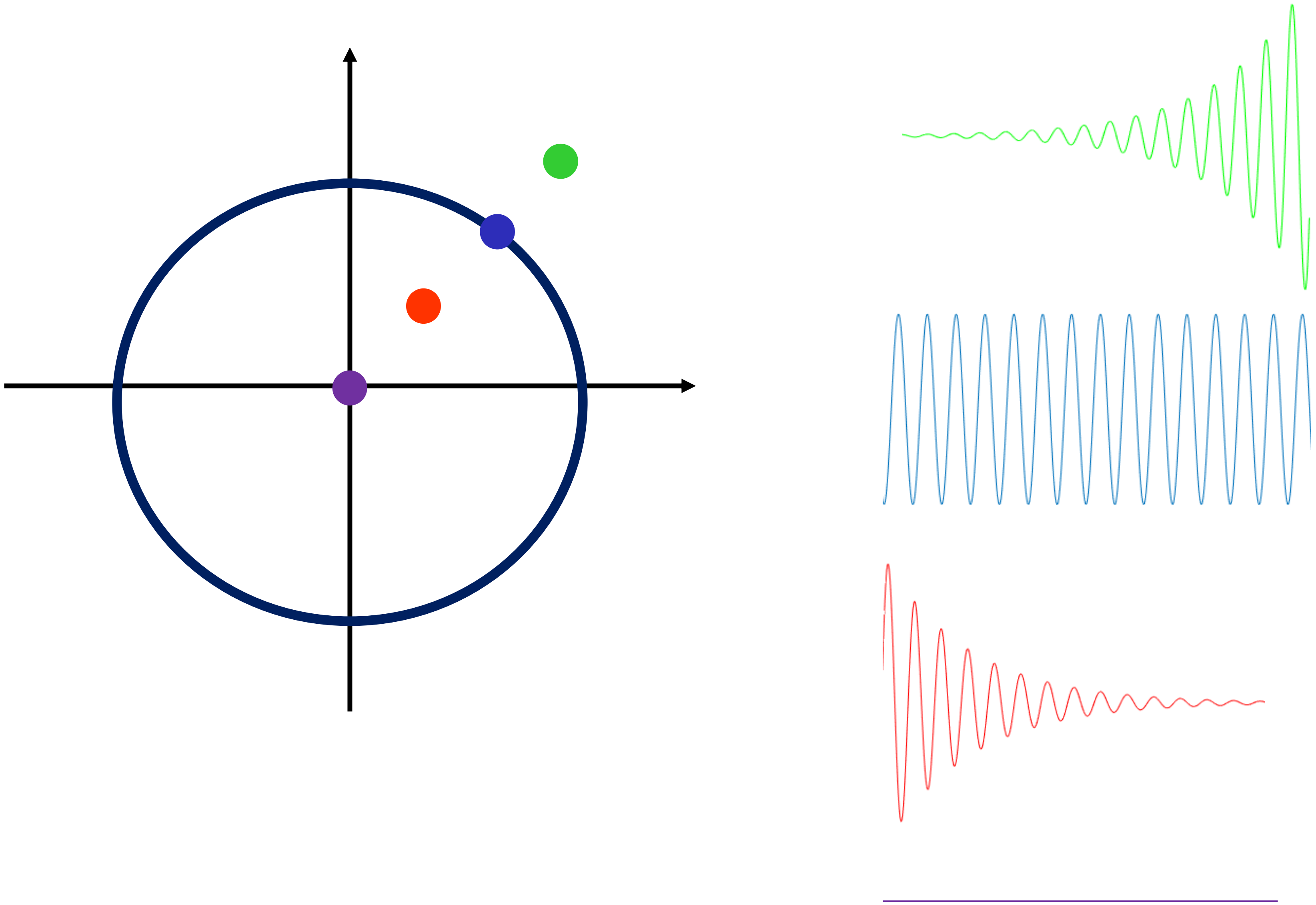
What is the DC signal?

$s_n = 1^n \ = \ \ldots +1 +1 +1 +1 \ldots$

What is the Nyquist signal?

$s_n = (-1)^n \ = \ \ldots -1 +1 -1 +1 \ldots$

Some more signals on the z plane



What's the connection with z?

We have used the letter z before -

for the time advance/delay operator

Is there a connection?

If we already know the zT of some signal x_n is X(z)

do we need to recalculate to find the zT of $\hat{\mathbf{z}}^{-1}$ x ?

No!

$$\sum_{n=-\infty}^{\infty} x_{n-1} z^{-n} = \sum_{n=-\infty}^{\infty} x_n z^{-(n+1)}$$

$$= z^{-1} \sum_{n=-\infty}^{\infty} x_n z^{-n} = z^{-1} \mathrm{zT}(x_n)$$

So $$\mathrm{zT}(\hat{\mathbf{z}}^{-1}\, x) = z^{-1}\, \mathrm{zT}(x)$$

delay operator **complex number**

What can you say about zT ($\hat{\mathbf{z}}$ x) ?

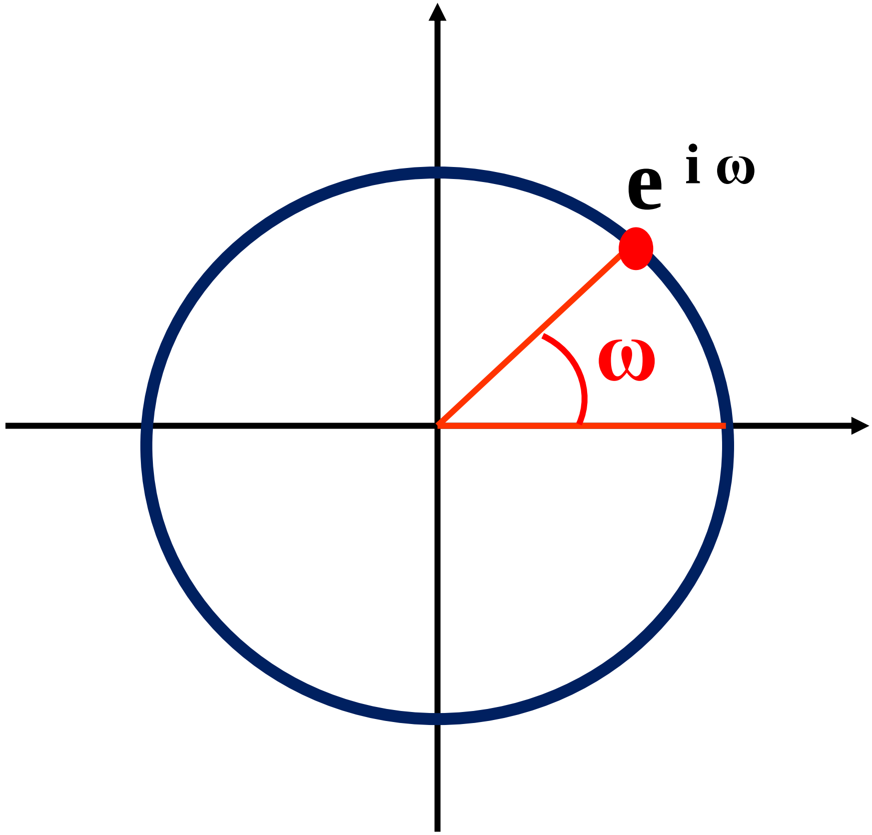
What's the connection with the DFT?

All complex numbers on the unit circle are of the form $e^{i \omega t}$

(since all complex numbers are $r e^{i \omega t}$ and here r=1)

So the zT on the unit circle is exactly the FT

$$S(z)\Big|_{z=e^{i\omega}} = \sum_{n=-\infty}^{\infty} s_n z^{-n} = \sum_{n=-\infty}^{\infty} s_n e^{-i\omega n}$$



If we look at points W_N we get exactly the DFT

Another riddle

Here is a proof that -1 = ∞

How much is $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$?

We all remember that S = 2 ! How do we prove this?

$S = 1 + \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ ...\right) = 1 \ + \frac{1}{2}S$

So $\frac{1}{2}S$ = 1 and so S = 2 !

Now, how much is $S = 1 + 2 + 4 + 8 + 16 + \ldots$?

I am sure you agree that $S = \infty$!

But using the same trick

$S = 1 + 2(1 + 2 + 4 + 8 + 16 + \ ...) = 1 \ + 2S$

So S = -1 !

What's wrong???

The *distributive law* only holds if the sum converges

RoC

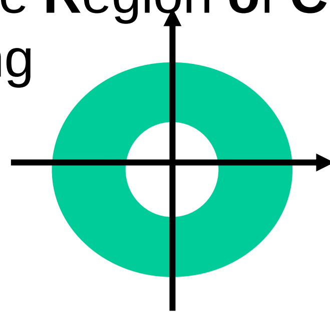
The zT is an infinite series

and we can get into trouble when it doesn't converge!

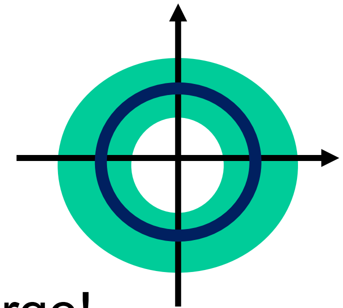
It turns out that the **R**egion **o**f **C**onvergence of the zT

is always a ring

General case

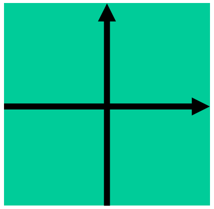


DFT converges if

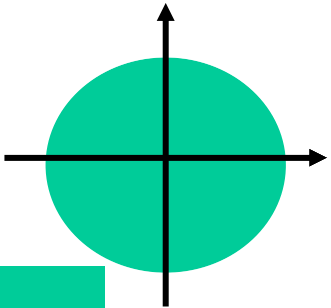


Sometimes we must use zT since DFT doesn't converge!

Special case : R1=0 R2=∞ RoC = entire plane



Special case : R1=0 R2<∞ RoC = circle



Special case : R1>0 R2=∞ RoC = outside of a circle